

Intertwining operators among modules for affine Lie algebra and lattice vertex operator algebras which respect integral forms

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Abstract

We define an integral intertwining operator among modules for a vertex operator algebra to be an intertwining operator which respects integral forms in the modules, and we show that an intertwining operator is integral if it is integral when restricted to generators of the integral forms in the modules. We apply this result to classify integral intertwining operators which respect certain natural integral forms in modules for affine Lie algebra and lattice vertex operator algebras.

1 Introduction

Integral forms for lattice vertex algebras were first introduced by Borcherds in [B] and were also studied in [P]. They have been used in the modular moonshine program initiated by Borcherds and Ryba ([R], [BR1], [BR2], [GL]). Recently, integral forms in vertex algebras related to lattice vertex algebras, including automorphism fixed points and the moonshine module V^\natural , have been constructed in [DG] and [GL], and their automorphism groups have been studied. The representation theory of integral vertex algebras based on affine Lie algebras and lattices was studied in [M2]: in particular, integral forms were constructed in both vertex algebras and their modules. It was also determined that the graded \mathbb{Z} -duals of integral forms in modules for these algebras form natural integral forms in the contragredient modules.

In the present paper, we continue to study the representation theory of integral vertex operator algebras in general, and in particular the representation theory of integral affine Lie algebra and lattice vertex operator algebras, by showing when intertwining operators among modules respect integral forms. Intertwining operators play an essential role in the representation theory of vertex operator algebras. For instance associativity of intertwining operators ([H1], [HLZ3]) and modular invariance for traces of products of intertwining operators ([H2]) are crucial for showing that modules for certain vertex operator algebras form modular tensor categories ([H3]; see also the review article [HL2]). Thus it is natural to consider intertwining operators among modules for vertex algebras over \mathbb{Z} .

Since also vertex algebras over \mathbb{Z} lead to vertex algebras over finite fields (and more generally over arbitrary fields of prime characteristic) through reducing structure constants mod a prime p , intertwining operators among modules for vertex algebras over \mathbb{Z} lead to

intertwining operators among modules for vertex algebras over fields of characteristic p . Intertwining operators among modules for the Virasoro vertex operator algebra $L(\frac{1}{2}, 0)$ over fields of odd prime characteristic have already been studied in [DR]. The present paper will have immediate application to intertwining operators among modules for affine Lie algebra and lattice vertex operator algebras in prime characteristic.

If V is a vertex operator algebra and $W^{(i)}$ for $i = 1, 2, 3$ are a triple of V -modules, then an intertwining operator of type $\left(\begin{smallmatrix} W^{(3)} \\ W^{(1)} W^{(2)} \end{smallmatrix}\right)$ is a linear map

$$\mathcal{Y} : W^{(1)} \otimes W^{(2)} \rightarrow W^{(3)}\{x\}$$

that satisfies a lower truncation axiom, an $L(-1)$ -derivative property, and a Jacobi identity. Here $W^{(3)}\{x\}$ represents formal series with arbitrary complex powers and coefficients in $W^{(3)}$. If V and the modules $W^{(i)}$ have integral forms $V_{\mathbb{Z}}$ and $W_{\mathbb{Z}}^{(i)}$, respectively, then it is natural to consider which intertwining operators among these modules respect the integral forms. In particular, it is natural to say that an intertwining operator of type $\left(\begin{smallmatrix} W^{(3)} \\ W^{(1)} W^{(2)} \end{smallmatrix}\right)$ is integral with respect to the integral forms $W_{\mathbb{Z}}^{(i)}$ if

$$\mathcal{Y}(w_{(1)}, x)w_{(2)} \in W_{\mathbb{Z}}^{(3)}\{x\} \quad (1.1)$$

for $w_{(1)} \in W_{\mathbb{Z}}^{(1)}$ and $w_{(2)} \in W_{\mathbb{Z}}^{(2)}$.

The main general result of this paper is that in order to check that an intertwining operator of type $\left(\begin{smallmatrix} W^{(3)} \\ W^{(1)} W^{(2)} \end{smallmatrix}\right)$ is integral, it is enough to check that (1.1) holds when $w_{(1)}$ and $w_{(2)}$ come from generating sets of $W_{\mathbb{Z}}^{(1)}$ and $W_{\mathbb{Z}}^{(2)}$, respectively. This result fits with the general philosophy of [M2] that integral forms of vertex operator algebras and modules are often best studied using generating sets. In the remainder of the paper, we apply the general theorem to vertex operator algebras coming from affine Lie algebras and lattices; natural integral forms in modules for these algebras were obtained in [M2].

Interestingly, the examples studied in this paper suggest another notion of integrality for intertwining operators different from (1.1). Specifically, suppose that V is an affine Lie algebra or lattice vertex operator algebra with modules $W^{(i)}$ for $i = 1, 2, 3$, and suppose that $V_{\mathbb{Z}}$ and $W_{\mathbb{Z}}^{(i)}$ are the integral forms in V and $W^{(i)}$, respectively, that were constructed in [M2]. Then we are able to find and classify non-zero intertwining operators which are integral with respect to $W_{\mathbb{Z}}^{(1)}$, $W_{\mathbb{Z}}^{(2)}$, and $((W^{(3)})'_{\mathbb{Z}})'$. Here $((W^{(3)})'_{\mathbb{Z}})'$ is the graded \mathbb{Z} -dual of the natural integral form in the contragredient module $(W^{(3)})'$; this is an integral form of $W^{(3)}$ which is often larger than $W_{\mathbb{Z}}^{(3)}$. In other words, we are able to classify the intertwining operators of type $\left(\begin{smallmatrix} W^{(3)} \\ W^{(1)} W^{(2)} \end{smallmatrix}\right)$ which satisfy

$$\langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)} \rangle \in \mathbb{Z}\{x\} \quad (1.2)$$

when $w_{(1)}$, $w_{(2)}$, and $w'_{(3)}$ come from the natural integral forms of $W^{(1)}$, $W^{(2)}$, and $(W^{(3)})'$, respectively. This suggests that (1.2) is a more fundamental integrality condition than (1.1), a notion supported by the fact that (1.2) corresponds to an integrality condition on physically relevant correlation functions in conformal field theory.

The remainder of this paper is structured as follows. In the next section we recall the notions of integral form in a vertex operator algebra and in a module for a vertex operator

algebra from [M2] and recall some results on integral forms in contragredient modules. In Section 3, we recall the definition of intertwining operator among modules for a vertex operator algebra and define what it means for an intertwining operator to be integral with respect to integral forms in the modules. We also prove our general result that an intertwining operator is integral if it is integral when restricted to generators. Finally, we derive a way to use intertwining operators to identify a module with its contragredient; this will be used in the last section when we study integral intertwining operators for lattice vertex algebras.

In Section 4 we recall the construction of the level ℓ generalized Verma module vertex operator algebra $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ based on an affine Lie algebra $\widehat{\mathfrak{g}}$, where \mathfrak{g} is a finite-dimensional simple complex Lie algebra. We also recall the classification of irreducible $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules, including the irreducible quotient $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ of $V_{\widehat{\mathfrak{g}}}(\ell, 0)$. We recall from [FZ] (and [Li2]) the characterization of intertwining operators among $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules for nonnegative integral level ℓ . In Section 5, we recall the natural integral forms in $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ - and $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules from [M2] and classify integral intertwining operators among $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules when ℓ is a nonnegative integer.

In Section 6, we recall the construction of the vertex algebra V_L and its modules from an even nondegenerate lattice L and recall from [DL] the construction of intertwining operators among V_L -modules. In Section 7, we recall from [M2] the natural integral form in a V_L -module and use nondegenerate invariant bilinear pairings to explicitly describe the graded \mathbb{Z} -dual of the natural integral form of a V_L -module. Using this, we classify integral intertwining operators among V_L -modules.

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2 Integral forms in vertex operator algebras and modules

Throughout this paper, we will use standard terminology and notation from the theory of vertex operator algebras, sometimes without comment. In particular, we will use the notion of vertex algebra as defined in [B] and the notion of vertex operator algebra as defined in [FLM] (see also [LL]). We will also use the notion of module for a vertex (operator) algebra as defined in [LL] (where modules for a vertex operator algebra have a conformal weight grading by \mathbb{C}). We recall that a vertex operator algebra and its modules admit representations of the Virasoro Lie algebra, and as usual we use $L(n)$ to denote the action of Virasoro algebra operators on a vertex operator algebra and its modules.

Vertex algebras over \mathbb{Z} make sense, and we refer to them as vertex rings. As in [M2], we define an integral form in a vertex operator algebra as follows:

Definition 2.1. *An integral form of a vertex operator algebra V is a vertex subring $V_{\mathbb{Z}} \subseteq V$ that is an integral form of V as a vector space and that is compatible with the conformal*

weight grading of V :

$$V_{\mathbb{Z}} = \coprod_{n \in \mathbb{Z}} V_n \cap V_{\mathbb{Z}},$$

where V_n is the weight space with $L(0)$ -eigenvalue n .

Remark 2.2. Equivalently, an integral form $V_{\mathbb{Z}}$ of V is the \mathbb{Z} -span of a basis for V which contains the vacuum $\mathbf{1}$, is closed under vertex algebra products, and is compatible with the conformal weight gradation.

Remark 2.3. In [DG], the definition of an integral form $V_{\mathbb{Z}}$ of V is somewhat different: compatibility with the weight gradation is replaced by the requirement that an integral multiple of the conformal vector ω be in $V_{\mathbb{Z}}$. In the examples studied in this paper, both conditions hold.

Once we have fixed an integral form $V_{\mathbb{Z}}$ of a vertex operator algebra V , it is easy to define the notion of an integral form in a V -module:

Definition 2.4. An integral form in a V -module W is a $V_{\mathbb{Z}}$ -submodule $W_{\mathbb{Z}} \subseteq W$ that is an integral form of W as a vector space and that is compatible with the conformal weight grading of W :

$$W_{\mathbb{Z}} = \coprod_{h \in \mathbb{C}} W_h \cap W_{\mathbb{Z}},$$

where W_h is the weight space with $L(0)$ -eigenvalue h .

Remark 2.5. Equivalently, an integral form $W_{\mathbb{Z}}$ of W is the \mathbb{Z} -span of a basis for W which is preserved by vertex operators from $V_{\mathbb{Z}}$ and is compatible with the conformal weight gradation.

Remark 2.6. Note that the notion of an integral form of W depends on the integral form $V_{\mathbb{Z}}$ of V that is used.

Here we note that our approach to studying integral forms in vertex operator algebras and their modules uses generating sets. If $V_{\mathbb{Z}}$ is an integral form of a vertex operator algebra V , we say that $S \subseteq V_{\mathbb{Z}}$ generates $V_{\mathbb{Z}}$ if $V_{\mathbb{Z}}$ is the smallest vertex subring of V containing S . Similarly, if W is a V -module with integral form $W_{\mathbb{Z}}$, we say that $T \subseteq W_{\mathbb{Z}}$ generates $W_{\mathbb{Z}}$ if $W_{\mathbb{Z}}$ is the smallest $V_{\mathbb{Z}}$ -submodule of W containing T .

For our study of intertwining operators in this paper, we will also need to use contragredients of modules for a vertex operator algebra V . Recall from [FHL] that if W is a V -module, the contragredient V -module W' is the graded dual

$$W' = \coprod_{h \in \mathbb{C}} W_h^*$$

equipped with the vertex operator defined by

$$\langle Y_{W'}(v, x)w', w \rangle = \langle w', Y_W^o(v, x)w \rangle$$

for $v \in V$, $w \in W$, and $w' \in W'$. Here Y_W^o denotes the *opposite vertex operator*

$$Y_W^o(v, x) = Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}). \quad (2.1)$$

Recall also from [FHL] that for any V -module W , $W \cong (W')'$, so that it makes sense to refer to contragredient pairs of V -modules.

If $V_{\mathbb{Z}}$ is an integral form of V and W is a V -module with integral form $W_{\mathbb{Z}}$, then the graded \mathbb{Z} -dual of $W_{\mathbb{Z}}$ defined by

$$W'_{\mathbb{Z}} = \{w' \in W' \mid \langle w', w \rangle \in \mathbb{Z} \text{ for } w \in W_{\mathbb{Z}}\}$$

is an integral form of W' as a vector space. The following two propositions from [M2] (see also Lemma 6.1, Lemma 6.2, and Remark 6.3 in [DG]) show when $W'_{\mathbb{Z}}$ is a $V_{\mathbb{Z}}$ -module:

Proposition 2.7. *Suppose $V_{\mathbb{Z}}$ is preserved by $\frac{L(1)^n}{n!}$ for $n \geq 0$. Then $W'_{\mathbb{Z}}$ is preserved by the action of $V_{\mathbb{Z}}$.*

Proposition 2.8. *If $V_{\mathbb{Z}}$ is generated by vectors v such that $L(1)v = 0$, then $V_{\mathbb{Z}}$ is preserved by $\frac{L(1)^n}{n!}$ for $n \geq 0$.*

Remark 2.9. If an integral form $V_{\mathbb{Z}}$ is invariant under the operators $\frac{L(1)^n}{n!}$ for $n \geq 0$, then in fact $V_{\mathbb{Z}}$ is invariant under the integral form of the universal enveloping algebra of $\mathfrak{sl}_2 = \text{span}\{L(-1), L(0), L(1)\}$ with basis given by the ordered products

$$\frac{L(-1)^k}{k!} \cdot \binom{L(0)}{m} \cdot \frac{L(1)^n}{n!}$$

where $k, m, n \geq 0$. Note that $V_{\mathbb{Z}}$ is automatically invariant under the operators $\frac{L(-1)^k}{k!}$ because $e^{xL(-1)}v = Y(v, x)\mathbf{1}$ for any $v \in V_{\mathbb{Z}}$ and $V_{\mathbb{Z}}$ is invariant under the operators $\binom{L(0)}{m}$ by compatibility of $V_{\mathbb{Z}}$ with the weight gradation. But note that we cannot expect all of these operators to preserve integral forms of V -modules because for instance V -modules may not be graded by integers.

Sometimes a vertex operator algebra V is equivalent as a V -module to its contragredient V' . From [FHL], this is the case if and only if V has a nondegenerate bilinear form (\cdot, \cdot) that is *invariant* in the sense that

$$(Y(u, x)v, w) = (v, Y^o(u, x)w)$$

for all $u, v, w \in V$. From [Lil], we know that the space of (not necessarily nondegenerate) invariant bilinear forms on V is linearly isomorphic to $V_0/L(1)V_1$, where as usual V_n for $n \in \mathbb{Z}$ represents the weight space with $L(0)$ -eigenvalue n . If V is a simple vertex operator algebra, then any non-zero invariant bilinear form is necessarily nondegenerate.

3 Integral intertwining operators

For any vector space or \mathbb{Z} -module V , let $V\{x\}$ denote the space of formal series with complex powers and coefficients in V . We recall the definition of intertwining operator among a triple of modules for a vertex operator algebra (see for instance [FHL]):

Definition 3.1. Suppose V is a vertex operator algebra and $W^{(1)}$, $W^{(2)}$ and $W^{(3)}$ are V -modules. An intertwining operator of type $\left(\begin{smallmatrix} W^{(3)} \\ W^{(1)} W^{(2)} \end{smallmatrix}\right)$ is a linear map

$$\begin{aligned} \mathcal{Y} : W^{(1)} \otimes W^{(2)} &\rightarrow W^{(3)}\{x\}, \\ w_{(1)} \otimes w_{(2)} &\mapsto \mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n \in \mathbb{C}} (w_{(1)})_n w_{(2)} x^{-n-1} \in W^{(3)}\{x\} \end{aligned}$$

satisfying the following conditions:

1. Lower truncation: For any $w_{(1)} \in W^{(1)}$, $w_{(2)} \in W^{(2)}$ and $n \in \mathbb{C}$,

$$(w_{(1)})_{n+m} w_{(2)} = 0 \quad \text{for } m \in \mathbb{N} \text{ sufficiently large.}$$

2. The Jacobi identity:

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_{W^{(3)}}(v, x_1) \mathcal{Y}(w_{(1)}, x_2) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \mathcal{Y}(w_{(1)}, x_2) Y_{W^{(2)}}(v, x_1) \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \mathcal{Y}(Y_{W^{(1)}}(v, x_0) w_{(1)}, x_2) \end{aligned} \quad (3.1)$$

for $v \in V$ and $w_{(1)} \in W^{(1)}$.

3. The $L(-1)$ -derivative property: for any $w_{(1)} \in W^{(1)}$,

$$\mathcal{Y}(L(-1)w_{(1)}, x) = \frac{d}{dx} \mathcal{Y}(w_{(1)}, x). \quad (3.2)$$

Remark 3.2. We use $V_{W^{(1)}W^{(2)}}^{W^{(3)}}$ to denote the vector space of intertwining operators of type $\left(\begin{smallmatrix} W^{(3)} \\ W^{(1)} W^{(2)} \end{smallmatrix}\right)$, and the dimension of $V_{W^{(1)}W^{(2)}}^{W^{(3)}}$ is the *fusion rule* $N_{W^{(1)}W^{(2)}}^{W^{(3)}}$

Remark 3.3. Note that if W is a V -module, the vertex operator Y_W is an intertwining operator of type $\left(\begin{smallmatrix} W \\ V W \end{smallmatrix}\right)$. The Jacobi identity (3.1) with $v = \omega$ implies that for $w_{(1)} \in W^{(1)}$ and $w_{(2)} \in W^{(2)}$ both homogeneous,

$$\text{wt} (w_{(1)})_n w_{(2)} = \text{wt } w_{(1)} + \text{wt } w_{(2)} - n - 1 \quad (3.3)$$

for any $n \in \mathbb{C}$.

Remark 3.4. If $W^{(1)}$, $W^{(2)}$ and $W^{(3)}$ are irreducible V -modules, then there are complex numbers h_i for $i = 1, 2, 3$ such that the conformal weights of $W^{(i)}$ are contained in $h_i + \mathbb{N}$ for each i . If \mathcal{Y} is an intertwining operator of type $\binom{W^{(3)}}{W^{(1)} W^{(2)}}$ and we set

$$h = h_1 + h_2 - h_3,$$

then (3.3) implies that we can write

$$\mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n \in \mathbb{Z}} w_{(1)}(n)w_{(2)} x^{-n-h} \quad (3.4)$$

for any $w_{(1)} \in W^{(1)}$ and $w_{(2)} \in W^{(2)}$, where $w_{(1)}(n) = (w_{(1)})_{n+h-1}$ for $n \in \mathbb{Z}$. In particular, for $w_{(1)} \in W_{h_1}^{(1)}$ and $W_{h_2}^{(2)}$, note that $w_{(1)}(0)w_{(2)} \in W_{h_3}^{(3)}$.

The Jacobi identity for intertwining operators, like the Jacobi identity for vertex algebras and modules, implies commutator and iterate formulas. We will in particular need the iterate formula: for any $v \in V$, $w_{(1)} \in W^{(1)}$, and $n \in \mathbb{Z}$,

$$\begin{aligned} \mathcal{Y}(v_n w_{(1)}, x_2) &= \text{Res}_{x_1} (x_1 - x_2)^n Y_{W^{(3)}}(v, x_1) \mathcal{Y}(w_{(1)}, x_2) \\ &\quad - \text{Res}_{x_1} (-x_2 + x_1)^n \mathcal{Y}(w_{(1)}, x_2) Y_{W^{(2)}}(v, x_1). \end{aligned} \quad (3.5)$$

We will also need weak commutativity for intertwining operators, whose proof is exactly the same as the proof of weak commutativity for algebras and modules (Propositions 3.2.1 and 4.2.1 in [LL]):

Proposition 3.5. *Suppose $W^{(1)}$, $W^{(2)}$ and $W^{(3)}$ are V -modules and $\mathcal{Y} \in V_{W^{(1)}W^{(2)}}^{W^{(3)}}$. Then for any positive integer k such that $v_n w_{(1)} = 0$ for $n \geq k$,*

$$(x_1 - x_2)^k Y_{W^{(3)}}(v, x_1) \mathcal{Y}(w_{(1)}, x_2) = (x_1 - x_2)^k \mathcal{Y}(w_{(1)}, x_2) Y_{W^{(2)}}(v, x_1).$$

Proof. Multiply the Jacobi identity (3.1) by x_0^k and extract the coefficient of x_0^{-1} , obtaining

$$\begin{aligned} (x_1 - x_2)^k Y_{W^{(3)}}(v, x_1) \mathcal{Y}(w_{(1)}, x_2) &- (x_1 - x_2)^k \mathcal{Y}(w_{(1)}, x_2) Y_{W^{(2)}}(v, x_1) \\ &= \text{Res}_{x_0} x_0^k x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \mathcal{Y}(Y_{W^{(1)}}(v, x_0) w_{(1)}, x_2). \end{aligned}$$

Since $v_n w_{(1)} = 0$ for $n \geq k$, the right side contains no negative powers of x_0 , so the residue is 0. \square

We will need the following proposition, which is similar to Proposition 4.5.8 in [LL] and uses essentially the same proof:

Proposition 3.6. *Suppose $W^{(1)}$, $W^{(2)}$, and $W^{(3)}$ are V -modules and $\mathcal{Y} \in V_{W^{(1)}W^{(2)}}^{W^{(3)}}$. Then for any $v \in V$, $w_{(1)} \in W^{(1)}$, and $w_{(2)} \in W^{(2)}$, and for any $p \in \mathbb{C}$ and $q \in \mathbb{Z}$, $(w_{(1)})_p v_q w_{(2)}$ is an integral linear combination of terms of the form $v_r (w_{(1)})_s w_{(2)}$. Specifically, let k and m be nonnegative integers such that $v_n w_{(1)} = 0$ for $n \geq k$ and $v_n w_{(2)} = 0$ for $n > m + q$. Then*

$$(w_{(1)})_p v_q w_{(2)} = \sum_{i=0}^m \sum_{j=0}^k (-1)^{i+j} \binom{-k}{i} \binom{k}{j} v_{q+i+j} (w_{(1)})_{p-i-j} w_{(2)}.$$

Proof. From weak commutativity, we have

$$\begin{aligned}
(w_{(1)})_p v_q w_{(2)} &= \text{Res}_{x_1} \text{Res}_{x_2} x_1^q x_2^p \mathcal{Y}(w_{(1)}, x_2) Y_{W^{(2)}}(v, x_1) w_{(2)} \\
&= \text{Res}_{x_1} \text{Res}_{x_2} x_1^q x_2^p (-x_2 + x_1)^{-k} [(x_1 - x_2)^k \mathcal{Y}(w_{(1)}, x_2) Y_{W^{(2)}}(v, x_1) w_{(2)}] \\
&= \text{Res}_{x_1} \text{Res}_{x_2} x_1^q x_2^p (-x_2 + x_1)^{-k} [(x_1 - x_2)^k Y_{W^{(3)}}(v, x_1) \mathcal{Y}(w_{(1)}, x_2) w_{(2)}]. \quad (3.6)
\end{aligned}$$

These formal expressions are well defined because $Y_{W^{(2)}}(v, x_1) w_{(2)}$ is lower-truncated, but we cannot remove the brackets from the last expression in (3.6). We observe that the term in brackets in (3.6) may be written explicitly as

$$\sum_{j \in \mathbb{N}, m \in \mathbb{Z}, n \in \mathbb{C}} (-1)^j \binom{k}{j} v_m(w_{(1)})_n w_{(2)} x_1^{k-j-m-1} x_2^{j-n-1}.$$

Meanwhile, only a finite truncation of $x_1^q x_2^p (-x_2 + x_1)^{-k}$ contributes to the residue since $Y_{W^{(2)}}(v, x_1) w_{(2)}$ is lower-truncated. In particular, the lowest possible integral power in $x_1^q Y_{W^{(2)}}(v, x_1) w_{(2)}$ with a non-zero coefficient is x_2^{-m-1} by definition of m . Thus we can take

$$\sum_{i=0}^m (-1)^{k+i} \binom{-k}{i} x_1^{q+i} x_2^{p-k-i}$$

as our truncation of $x_1^q x_2^p (-x_2 + x_1)^{-k}$. Then

$$\begin{aligned}
&(w_{(1)})_p v_q w_{(2)} \\
&= \text{Res}_{x_1, x_2} \sum_{i, j, m, n} (-1)^{k+i+j} \binom{-k}{i} \binom{k}{j} v_m(w_{(1)})_n w_{(2)} x_1^{q+k+i-j-m-1} x_2^{p-k-i+j-n-1} \\
&= \sum_{i=0}^m \sum_{j=0}^k (-1)^{k+i+j} \binom{-k}{i} \binom{k}{j} v_{q+k+i-j}(w_{(1)})_{p-k-i+j} w_{(2)} \\
&= \sum_{i=0}^m \sum_{j=0}^k (-1)^{i+j} \binom{-k}{i} \binom{k}{j} v_{q+i+j}(w_{(1)})_{p-i-j} w_{(2)},
\end{aligned}$$

where we have changed j to $k - j$ in the last equality. \square

The Jacobi identity for intertwining operators makes sense in the vertex ring context for the same reasons that the Jacobi identity for algebras and modules does. However, the $L(-1)$ -derivative property may not make sense. If $V_{\mathbb{Z}}$ is a vertex ring without a conformal vector ω , then there may be no $L(-1)$ operator on $V_{\mathbb{Z}}$ -modules. Additionally, since intertwining operators involve complex powers of x , the coefficients of the derivative of an intertwining operator may not make sense as maps from a $V_{\mathbb{Z}}$ -module into a $V_{\mathbb{Z}}$ -module.

However, suppose V is a vertex operator algebra with integral form $V_{\mathbb{Z}}$ and $W^{(1)}$, $W^{(2)}$, and $W^{(3)}$ are V -modules with integral forms $W_{\mathbb{Z}}^{(1)}$, $W_{\mathbb{Z}}^{(2)}$, and $W_{\mathbb{Z}}^{(3)}$, respectively:

Definition 3.7. An intertwining operator $\mathcal{Y} \in V_{W^{(1)}W^{(2)}}^{W^{(3)}}$ is integral with respect to $W_{\mathbb{Z}}^{(1)}$, $W_{\mathbb{Z}}^{(2)}$, and $W_{\mathbb{Z}}^{(3)}$ if for any $w_{(1)} \in W_{\mathbb{Z}}^{(1)}$ and $w_{(2)} \in W_{\mathbb{Z}}^{(2)}$,

$$\mathcal{Y}(w_{(1)}, x) w_{(2)} \in W_{\mathbb{Z}}^{(3)} \{x\}.$$

Remark 3.8. Note that whether an intertwining operator is integral or not will generally depend on the integral forms used for the three V -modules.

The main result of this section is the following theorem which reduces the problem of showing that an intertwining operator is integral to the problem of showing that it is integral when restricted to generators of $W^{(1)}$ and $W^{(2)}$:

Theorem 3.9. *Suppose V is a vertex operator algebra with integral form $V_{\mathbb{Z}}$ and $W^{(1)}$, $W^{(2)}$, and $W^{(3)}$ are V -modules with integral forms $W_{\mathbb{Z}}^{(1)}$, $W_{\mathbb{Z}}^{(2)}$, and $W_{\mathbb{Z}}^{(3)}$, respectively. Moreover, suppose $T^{(1)}$ and $T^{(2)}$ are generating sets for $W_{\mathbb{Z}}^{(1)}$ and $W_{\mathbb{Z}}^{(2)}$, respectively. If $\mathcal{Y} \in V_{W^{(1)}W^{(2)}}^{W^{(3)}}$ satisfies*

$$\mathcal{Y}(t_{(1)}, x)t_{(2)} \in W_{\mathbb{Z}}^{(3)}\{x\}$$

for all $t_{(1)} \in T^{(1)}$, $t_{(2)} \in T^{(2)}$, then \mathcal{Y} is integral with respect to $W_{\mathbb{Z}}^{(1)}$, $W_{\mathbb{Z}}^{(2)}$, and $W_{\mathbb{Z}}^{(3)}$.

Proof. First let $U_{\mathbb{Z}}^{(2)}$ be the sublattice of $W_{\mathbb{Z}}^{(2)}$ consisting of vectors $u_{(2)}$ such that

$$\mathcal{Y}(t_{(1)}, x)u_{(2)} \in W_{\mathbb{Z}}^{(3)}\{x\}$$

for all $t_{(1)} \in T^{(1)}$. Note that if $u_{(2)} \in U_{\mathbb{Z}}^{(2)}$, $t_{(1)} \in T^{(1)}$, and $v \in V_{\mathbb{Z}}$, Proposition 3.6 implies that for any $p \in \mathbb{C}$ and $q \in \mathbb{Z}$, $(t_{(1)})_p v_q u_{(2)}$ is an integral linear combination of terms of the form $v_r (t_{(1)})_s u_{(2)}$, and is thus in $W_{\mathbb{Z}}^{(3)}$. This means that

$$Y_{W^{(2)}}(v, x)u_{(2)} \in U_{\mathbb{Z}}^{(2)}[[x, x^{-1}]],$$

for any $v \in V_{\mathbb{Z}}$, $u_{(2)} \in U_{\mathbb{Z}}^{(2)}$, that is, $U_{\mathbb{Z}}^{(2)}$ is a $V_{\mathbb{Z}}$ -module. Since by hypothesis $U_{\mathbb{Z}}^{(2)}$ contains $T^{(2)}$ which generates $W_{\mathbb{Z}}^{(2)}$, $U_{\mathbb{Z}}^{(2)} = W_{\mathbb{Z}}^{(2)}$ and so

$$\mathcal{Y}(t_{(1)}, x)w_{(2)} \in W_{\mathbb{Z}}^{(3)}\{x\} \tag{3.7}$$

for any $t_{(1)} \in T^{(1)}$ and $w_{(2)} \in W_{\mathbb{Z}}^{(2)}$.

Now we define $U_{\mathbb{Z}}^{(1)}$ as the sublattice of $W_{\mathbb{Z}}^{(1)}$ consisting of all vectors $u_{(1)} \in W_{\mathbb{Z}}^{(1)}$ such that

$$\mathcal{Y}(u_{(1)}, x)w_{(2)} \in W_{\mathbb{Z}}^{(3)}\{x\}.$$

for all $w_{(2)} \in W_{\mathbb{Z}}^{(2)}$. In this case, $U_{\mathbb{Z}}^{(1)}$ is a $V_{\mathbb{Z}}$ -submodule of $W_{\mathbb{Z}}^{(1)}$ by the iterate formula for intertwining operators, (3.5). Since $U_{\mathbb{Z}}^{(1)}$ contains $T^{(1)}$ by (3.7), $U_{\mathbb{Z}}^{(1)} = W_{\mathbb{Z}}^{(1)}$, and we have shown that

$$\mathcal{Y}(w_{(1)}, x)w_{(2)} \in W_{\mathbb{Z}}^{(3)}\{x\}$$

for any $w_{(1)} \in W_{\mathbb{Z}}^{(1)}$ and $w_{(2)} \in W_{\mathbb{Z}}^{(2)}$, that is, \mathcal{Y} is integral with respect to $W_{\mathbb{Z}}^{(1)}$, $W_{\mathbb{Z}}^{(2)}$, and $W_{\mathbb{Z}}^{(3)}$. \square

Remark 3.10. Note the role of Proposition 3.6 in the proof of the theorem. In particular, the commutator formula for intertwining operators (obtained from the coefficient of x_0^{-1} in the Jacobi identity (3.1)) is not enough by itself without additional assumptions on the generating set for $W_{\mathbb{Z}}^{(1)}$.

We will apply Theorem 3.9 to classify integral intertwining operators among modules for affine Lie algebra and lattice vertex operator algebras in the following sections. In both examples, the graded \mathbb{Z} -dual of a module integral form will also play a role. In the case of modules for lattice vertex operator algebras in Section 7, we will need to realize the graded \mathbb{Z} -dual explicitly using intertwining operators and Proposition 3.12 below.

We need to recall the symmetries among spaces of intertwining operators from [FHL], [HL1], and [HLZ2]. First, if $W^{(1)}$, $W^{(2)}$ and $W^{(3)}$ are V -modules and \mathcal{Y} is an intertwining operator of type $\binom{W^{(3)}}{W^{(1)} W^{(2)}}$, then for any $r \in \mathbb{Z}$, there is an intertwining operator $\Omega_r(\mathcal{Y})$ of type $\binom{W^{(3)}}{W^{(2)} W^{(1)}}$ defined by

$$\Omega_r(\mathcal{Y})(w_{(2)}, x)w_{(1)} = e^{xL(-1)}\mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i}x)w_{(2)} \quad (3.8)$$

for $w_{(1)} \in W^{(1)}$ and $w_{(2)} \in W^{(2)}$. Moreover, for any $r \in \mathbb{Z}$ there is an intertwining operator $A_r(\mathcal{Y})$ of type $\binom{(W^{(2)})'}{W^{(1)} (W^{(3)})'}$ defined by

$$\langle A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)}, w_{(2)} \rangle_{W^{(2)}} = \langle w'_{(3)}, \mathcal{Y}_r^o(w_{(1)}, x)w_{(2)} \rangle_{W^{(3)}} \quad (3.9)$$

for $w_{(1)} \in W^{(1)}$, $w_{(2)} \in W^{(2)}$, and $w'_{(3)} \in (W^{(3)})'$, where

$$\mathcal{Y}_r^o(w_{(1)}, x)w_{(2)} = \mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi i L(0)}(x^{-L(0)})^2 w_{(1)}, x^{-1})w_{(2)}.$$

Then we have (see for example Propositions 3.44 and 3.46 in [HLZ2]):

Proposition 3.11. *For any $r \in \mathbb{Z}$, the map $\Omega_r : V_{W^{(1)}W^{(2)}}^{W^{(3)}} \rightarrow V_{W^{(2)}W^{(1)}}^{W^{(3)}}$ is a linear isomorphism with inverse Ω_{-r-1} . Moreover, the map $A_r : V_{W^{(1)}W^{(2)}}^{W^{(3)}} \rightarrow V_{W^{(1)}(W^{(3)})'}^{(W^{(2)})'}$ is a linear isomorphism with inverse A_{-r-1} for any $r \in \mathbb{Z}$.*

We say that a bilinear pairing (\cdot, \cdot) between V -modules $W^{(1)}$ and $W^{(2)}$ is *invariant* if

$$(Y_{W^{(1)}}(v, x)w_{(1)}, w_{(2)}) = (w_{(1)}, Y_{W^{(2)}}^o(v, x)w_{(2)})$$

for $v \in V$, $w_{(1)} \in W^{(1)}$, and $w_{(2)} \in W^{(2)}$. It is clear that if there is a nondegenerate invariant bilinear pairing between $W^{(1)}$ and $W^{(2)}$, then $W^{(1)}$ and $W^{(2)}$ form a contragredient pair. The next proposition, which is a minor generalization of Remark 2.9 in [Li1], shows how certain intertwining operators yield nondegenerate invariant pairings between modules:

Proposition 3.12. *Suppose V is a vertex operator algebra equipped with a nondegenerate invariant bilinear pairing $(\cdot, \cdot)_V$, and suppose $W^{(1)}$ and $W^{(2)}$ are V -modules. If \mathcal{Y} is an intertwining operator of type $\binom{V}{W^{(1)} W^{(2)}}$, then the bilinear pairing (\cdot, \cdot) between $W^{(1)}$ and $W^{(2)}$ given by*

$$(w_{(1)}, w_{(2)}) = \text{Res}_x (\mathbf{1}, \mathcal{Y}_0^o(w_{(1)}, e^{\pi i}x)e^{xL(1)}w_{(2)})_V \quad (3.10)$$

for $w_{(1)} \in W^{(1)}$ and $w_{(2)} \in W^{(2)}$ is invariant. Moreover, if $W^{(1)}$ and $W^{(2)}$ are irreducible and \mathcal{Y} is non-zero, then the pairing is nondegenerate and $W^{(1)}$ and $W^{(2)}$ form a contragredient pair.

Proof. We consider the intertwining operator $\mathcal{Y}' = \Omega_0(A_0(\mathcal{Y}))$ of type $(\binom{(W^{(2)})'}{V \cong V' W^{(1)}})$. The $L(-1)$ -derivative property (3.2) implies that $\mathcal{Y}'(\mathbf{1}, x)$ equals its constant term $\mathbf{1}_{-1}$. Moreover, the coefficient of x_0^{-1} in the Jacobi identity (3.1) implies that

$$Y_{(W^{(2)})'}(v, x_1)\mathbf{1}_{-1} - \mathbf{1}_{-1}Y_{W^{(1)}}(v, x_1) = 0$$

since $Y(v, x_0)\mathbf{1}$ has no negative powers of x_0 . Thus $\mathbf{1}_{-1} = \varphi_{\mathcal{Y}}$ is a V -homomorphism from $W^{(1)}$ to $(W^{(2)})'$ and we obtain a bilinear pairing (\cdot, \cdot) between $W^{(1)}$ and $W^{(2)}$ given by

$$(w_{(1)}, w_{(2)}) = \langle \varphi_{\mathcal{Y}}(w_{(1)}), w_{(2)} \rangle$$

for $w_{(1)} \in W^{(1)}$, $w_{(2)} \in W^{(2)}$, which is invariant because $\varphi_{\mathcal{Y}}$ is a homomorphism.

To show that the invariant pairing (\cdot, \cdot) is also given by (3.10), we calculate using the definitions of Ω_0 and A_0 from (3.8) and (3.9), and identifying $V \cong V'$ via $(\cdot, \cdot)_V$. For $w_{(1)} \in W^{(1)}$ and $w_{(2)} \in W^{(2)}$,

$$\begin{aligned} (w_{(1)}, w_{(2)}) &= \langle \varphi_{\mathcal{Y}}(w_{(1)}), w_{(2)} \rangle = \text{Res}_x x^{-1} \langle \Omega_0(A_0(\mathcal{Y}))(\mathbf{1}, x)w_{(1)}, w_{(2)} \rangle \\ &= \text{Res}_x x^{-1} \langle e^{xL(-1)}A_0(\mathcal{Y})(w_{(1)}, e^{\pi i}x)\mathbf{1}, w_{(2)} \rangle \\ &= \text{Res}_x x^{-1} \langle A_0(\mathcal{Y})(w_{(1)}, e^{\pi i}x)\mathbf{1}, e^{xL(1)}w_{(2)} \rangle \\ &= \text{Res}_x x^{-1} (\mathbf{1}, \mathcal{Y}_0^o(w_{(1)}, e^{\pi i}x)e^{xL(1)}w_{(2)})_V. \end{aligned}$$

This proves the first assertion of the proposition.

To prove the nondegeneracy of (\cdot, \cdot) when $W^{(1)}$, $W^{(2)}$ are irreducible and \mathcal{Y} is non-zero, it is enough to prove that $\varphi_{\mathcal{Y}} : W^{(1)} \rightarrow (W^{(2)})'$ is an isomorphism. Since $W^{(1)}$ and $W^{(2)}$ are irreducible, it suffices to prove that $\varphi_{\mathcal{Y}}$ is non-zero, that is, $\mathcal{Y}'(\mathbf{1}, x) \neq 0$. But if $\mathcal{Y}'(\mathbf{1}, x) = 0$, then the creation property for vertex operator algebras and the iterate formula (3.5) imply that for any $v \in V$,

$$\mathcal{Y}'(v, x) = \mathcal{Y}'(v_{-1}\mathbf{1}, x) = 0.$$

This is a contradiction because by Proposition 3.11, \mathcal{Y}' is non-zero if \mathcal{Y} is. \square

4 Affine Lie algebra vertex operator algebras, their modules, and intertwining operators

In this section, we shall recall what we need from the representation theory of vertex operator algebras based on affine Lie algebras. We fix a finite-dimensional complex simple Lie algebra \mathfrak{g} , and let \mathfrak{h} denote a Cartan subalgebra of \mathfrak{g} . There is a unique up to scale nondegenerate invariant bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{g} , which is nondegenerate on \mathfrak{h} and induces a bilinear form on \mathfrak{h}^* . We shall normalize the form $\langle \cdot, \cdot \rangle$ on \mathfrak{g} so that

$$\langle \alpha, \alpha \rangle = 2$$

for long roots $\alpha \in \mathfrak{h}^*$. Irreducible finite-dimensional \mathfrak{g} -modules are in one-to-one correspondence with dominant integral weights $\lambda \in \mathfrak{h}^*$.

We now recall the affine Lie algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$$

where \mathbf{k} is central and all other brackets are determined by

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\langle a, b \rangle \delta_{m+n,0} \mathbf{k}$$

for $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. Using the notation of [LL] Chapter 6, for any finite-dimensional \mathfrak{g} -module U and any level $\ell \in \mathbb{C}$, we have the generalized Verma module $V_{\widehat{\mathfrak{g}}}(\ell, U)$ on which \mathbf{k} acts as the scalar ℓ . The generalized Verma module $V_{\widehat{\mathfrak{g}}}(\ell, U)$ has a unique irreducible quotient $L_{\widehat{\mathfrak{g}}}(\ell, U)$. The $\widehat{\mathfrak{g}}$ -modules $V_{\widehat{\mathfrak{g}}}(\ell, U)$ and $L_{\widehat{\mathfrak{g}}}(\ell, U)$ are linearly spanned by vectors of the form

$$a_1(-n_1) \cdots a_k(-n_k)u \quad (4.1)$$

for $a_i \in \mathfrak{g}$, $n_i > 0$. Here we use $a(n)$ for $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$ to denote the action of $a \otimes t^n$ on a $\widehat{\mathfrak{g}}$ -module.

In the case $U = \mathbb{C}\mathbf{1}$, the trivial one-dimensional \mathfrak{g} -module, $V_{\widehat{\mathfrak{g}}}(\ell, U) = V_{\widehat{\mathfrak{g}}}(\ell, 0)$ is a vertex algebra, as is its irreducible quotient $L_{\widehat{\mathfrak{g}}}(\ell, 0)$. The vertex operator map is determined by

$$Y(a(-1)\mathbf{1}, x) = \sum_{n \in \mathbb{Z}} a(n)x^{-n-1} \quad (4.2)$$

for $a \in \mathfrak{g}$. When $\ell \neq -h$ where h is the dual Coxeter number of \mathfrak{g} , $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ and $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ are also vertex operator algebras.

Let L_{λ} denote the irreducible \mathfrak{g} -module with highest weight λ . Then for any level $\ell \in \mathbb{C}$, the irreducible $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules consist of the modules $L(\ell, L_{\lambda})$ for λ a dominant integral weight. Moreover, suppose θ is the highest root of \mathfrak{g} ; if ℓ is a nonnegative integer, the irreducible $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules are given by $L(\ell, L_{\lambda})$ where λ is a dominant integral weight satisfying $\langle \lambda, \theta \rangle \leq \ell$ ([FZ]; see also [LL]).

As is well known, the dual of an irreducible \mathfrak{g} -module L_{λ} is also an irreducible \mathfrak{g} -module with the action given by

$$\langle x \cdot v', v \rangle = -\langle v', x \cdot v \rangle$$

for $x \in \mathfrak{g}$, $v \in L_{\lambda}$, and $v' \in L_{\lambda}^*$. So we have $L_{\lambda}^* \cong L_{\lambda^*}$ for some dominant integral weight λ^* . In fact, $\lambda^* = -w(\lambda)$ where w is the element in the Weyl group of \mathfrak{g} of maximal length (see for example [Hu]). Then the contragredient of the $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ -module $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda})$ is isomorphic to $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda^*})$ ([FZ]).

From now on, we assume that ℓ is a nonnegative integer; suppose that $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})$, $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})$, and $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})$ are irreducible $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules, so that $\langle \lambda_i, \theta \rangle \leq \ell$ for $i = 1, 2, 3$. We recall from [FZ] (see also [Li2]) the classification of intertwining operators of type $\binom{L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})}{L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1}) L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})}$. First, let $A(L_{\widehat{\mathfrak{g}}}(\ell, 0))$ be the Zhu's algebra of $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ (see [Z] for the definition); from [FZ], as an associative algebra,

$$A(L_{\widehat{\mathfrak{g}}}(\ell, 0)) \cong U(\mathfrak{g}) / \langle x_{\theta}^{\ell+1} \rangle,$$

where x_{θ} is a root vector for the longest root θ of \mathfrak{g} . We also need the $A(L_{\widehat{\mathfrak{g}}}(\ell, 0))$ -bimodule $A(L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1}))$, which from [FZ] is given by

$$A(L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})) \cong (L_{\lambda_1} \otimes U(\mathfrak{g})) / \langle v_{\lambda_1} \otimes x_{\theta}^{\ell - \langle \lambda_1, \theta \rangle + 1} \rangle,$$

where v_{λ_1} is a highest weight vector of L_{λ_1} and $\langle v_{\lambda_1} \otimes x_{\theta}^{\ell-\langle \lambda_1, \theta \rangle + 1} \rangle$ indicates the subbimodule generated by the indicated element. The $A(L_{\widehat{\mathfrak{g}}}(\ell, 0)) \cong U(\mathfrak{g})/\langle x_{\theta}^{\ell+1} \rangle$ -bimodule structure on $(L_{\lambda_1} \otimes U(\mathfrak{g}))/\langle v_{\lambda_1} \otimes x_{\theta}^{\ell-\langle \lambda_1, \theta \rangle + 1} \rangle$ is induced by the following $U(\mathfrak{g})$ -bimodule structure on $L_{\lambda_1} \otimes U(\mathfrak{g})$:

$$x \cdot (v \otimes y) = (x \cdot v) \otimes y + v \otimes xy$$

for $x, y \in U(\mathfrak{g})$, $v \in L_{\lambda_1}$, and

$$(v \otimes y) \cdot x = v \otimes yx.$$

We also recall from [FZ] that the lowest weight spaces L_{λ_2} and L_{λ_3} of $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})$ and $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})$, respectively, are (left) $A(L_{\widehat{\mathfrak{g}}}(\ell, 0))$ -modules in the natural way.

The description of the space of intertwining operators of type $\left(\begin{smallmatrix} L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3}) \\ L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1}) \quad L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2}) \end{smallmatrix} \right)$ from Theorem 2.11 and Corollary 2.13 in [Li2] (see also Theorems 1.5.2 and 1.5.3 in [FZ]) is as follows:

$$V_{L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1}) L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})}^{L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})} \cong \text{Hom}_{A(L_{\widehat{\mathfrak{g}}}(\ell, 0))}(A(L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})) \otimes_{A(L_{\widehat{\mathfrak{g}}}(\ell, 0))} L_{\lambda_2}, L_{\lambda_3}).$$

This space of $A(L_{\widehat{\mathfrak{g}}}(\ell, 0))$ -homomorphisms can be described more usefully using the following lemma:

Lemma 4.1. *As (left) modules for $A(L_{\widehat{\mathfrak{g}}}(\ell, 0)) \cong U(\mathfrak{g})/\langle x_{\theta}^{\ell+1} \rangle$,*

$$A(L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})) \otimes_{A(L_{\widehat{\mathfrak{g}}}(\ell, 0))} L_{\lambda_2} \cong (L_{\lambda_1} \otimes L_{\lambda_2})/W,$$

with W the $U(\mathfrak{g})/\langle x_{\theta}^{\ell+1} \rangle$ -module generated by all vectors of the form $v_{\lambda_1} \otimes x_{\theta}^{\ell-\langle \lambda_1, \theta \rangle + 1} \cdot w$ for $w \in L_{\lambda_2}$.

Proof. We know that $A(L_{\widehat{\mathfrak{g}}}(\ell, \lambda_1)) \cong (L_{\lambda_1} \otimes U(\mathfrak{g}))/W'$, where W' is the subbimodule generated by $v_{\lambda_1} \otimes x_{\theta}^{\ell-\langle \lambda_1, \theta \rangle + 1}$. We first define a map

$$\Phi : (L_{\lambda_1} \otimes U(\mathfrak{g})) \otimes_{U(\mathfrak{g})} L_{\lambda_2} \rightarrow (L_{\lambda_1} \otimes L_{\lambda_2})/W$$

as follows: for $u = v \otimes x$ where $v \in L_{\lambda_1}$ and $x \in U(\mathfrak{g})$, and for $w \in L_{\lambda_2}$, we define

$$\Phi(u \otimes w) = v \otimes x \cdot w + W.$$

By the left $U(\mathfrak{g})$ -module structure on $(L_{\lambda_1} \otimes U(\mathfrak{g})) \otimes_{U(\mathfrak{g})} L_{\lambda_2}$ and the tensor product \mathfrak{g} -module structure on $(L_{\lambda_1} \otimes L_{\lambda_2})/W$, it is easy to see that Φ is a $U(\mathfrak{g})$ -homomorphism. Moreover, Φ induces a $U(\mathfrak{g})/\langle x_{\theta}^{\ell+1} \rangle$ -module homomorphism

$$\varphi : (L_{\lambda_1} \otimes U(\mathfrak{g}))/W' \otimes_{U(\mathfrak{g})/\langle x_{\theta}^{\ell+1} \rangle} L_{\lambda_2} \rightarrow (L_{\lambda_1} \otimes L_{\lambda_2})/W$$

because for $y, y' \in U(\mathfrak{g})$ and $w \in L_{\lambda_2}$,

$$\begin{aligned} \Phi(y \cdot (v_{\lambda_1} \otimes x_{\theta}^{\ell-\langle \lambda_1, \theta \rangle + 1}) \cdot y' \otimes w) \\ &= \Phi((y \cdot v_{\lambda_1}) \otimes x_{\theta}^{\ell-\langle \lambda_1, \theta \rangle + 1} y' \otimes w + v_{\lambda_1} \otimes y x_{\theta}^{\ell-\langle \lambda_1, \theta \rangle + 1} y' \otimes w) \\ &= (y \cdot v_{\lambda_1}) \otimes (x_{\theta}^{\ell-\langle \lambda_1, \theta \rangle + 1} y') \cdot w + v_{\lambda_1} \otimes (y x_{\theta}^{\ell-\langle \lambda_1, \theta \rangle + 1} y') \cdot w + W \\ &= y \cdot (v_{\lambda_1} \otimes (x_{\theta}^{\ell-\langle \lambda_1, \theta \rangle + 1} y') \cdot w) + W = 0. \end{aligned}$$

To obtain an inverse homomorphism, we define a $U(\mathfrak{g})$ -homomorphism

$$\Psi : L_{\lambda_1} \otimes L_{\lambda_2} \rightarrow (L_{\lambda_1} \otimes U(\mathfrak{g}))/W' \otimes_{U(\mathfrak{g})/\langle x_\theta^{\ell+1} \rangle} L_{\lambda_2}$$

by defining for $u \in L_{\lambda_1}$ and $w \in L_{\lambda_2}$:

$$\Psi(u \otimes w) = (u \otimes 1 + W') \otimes w.$$

The map Ψ induces a $U(\mathfrak{g})/\langle x_\theta^{\ell+1} \rangle$ -homomorphism

$$\psi : (L_{\lambda_1} \otimes L_{\lambda_2})/W \rightarrow (L_{\lambda_1} \otimes U(\mathfrak{g}))/W' \otimes_{U(\mathfrak{g})/\langle x_\theta^{\ell+1} \rangle} L_{\lambda_2}$$

because for $x \in U(\mathfrak{g})$ and $w \in L_{\lambda_2}$,

$$\begin{aligned} \Psi(x \cdot (v_{\lambda_1} \otimes x_\theta^{\ell-\langle \lambda_1, \theta \rangle + 1} \cdot w)) &= \Psi((x \cdot v_{\lambda_1}) \otimes x_\theta^{\ell-\langle \lambda_1, \theta \rangle + 1} \cdot w + v_{\lambda_1} \otimes (xx_\theta^{\ell-\langle \lambda_1, \theta \rangle + 1}) \cdot w) \\ &= ((x \cdot v_{\lambda_1}) \otimes 1 + W') \otimes x_\theta^{\ell-\langle \lambda_1, \theta \rangle + 1} \cdot w + (v_{\lambda_1} \otimes 1 + W') \otimes xx_\theta^{\ell-\langle \lambda_1, \theta \rangle + 1} \cdot w \\ &= ((x \cdot v_{\lambda_1}) \otimes x_\theta^{\ell-\langle \lambda_1, \theta \rangle + 1} + W') \otimes w + (v_{\lambda_1} \otimes xx_\theta^{\ell-\langle \lambda_1, \theta \rangle + 1} + W') \otimes w \\ &= (x \cdot (v_{\lambda_1} \otimes x_\theta^{\ell-\langle \lambda_1, \theta \rangle + 1}) + W') \otimes w = 0. \end{aligned}$$

From the definitions, it is easy to see that φ and ψ are inverses of each other, so they give $A(L_{\widehat{\mathfrak{g}}}(\ell, 0)) \cong U(\mathfrak{g})/\langle x_\theta^{\ell+1} \rangle$ -module isomorphisms. \square

From this lemma, we have (see Theorem 3.2.3 in [FZ]):

$$V_{L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1}) L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})}^{L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})} \cong \text{Hom}_{\mathfrak{g}}((L_{\lambda_1} \otimes L_{\lambda_2})/W, L_{\lambda_3}).$$

From [Li2] and [FZ], we can describe the isomorphism as follows. An intertwining operator \mathcal{Y} of type $\left(\begin{smallmatrix} L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3}) \\ L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1}) L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2}) \end{smallmatrix} \right)$ induces a \mathfrak{g} -homomorphism

$$\pi(\mathcal{Y}) : L_{\lambda_1} \otimes L_{\lambda_2} \rightarrow L_{\lambda_3}$$

given by

$$\pi(\mathcal{Y})(w_{(1)} \otimes w_{(2)}) = w_{(1)}(0)w_{(2)} \tag{4.3}$$

using the notation of (3.4), where $w_{(1)} \in L_{\lambda_1} \subseteq L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})$ and $w_{(2)} \in L_{\lambda_2} \subseteq L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})$. This homomorphism $\pi(\mathcal{Y})$ must equal 0 on W , so it induces a homomorphism, which we also call $\pi(\mathcal{Y})$, from $(L_{\lambda_1} \otimes L_{\lambda_2})/W$ to L_{λ_3} . Note that it is not trivial to show that $\mathcal{Y} \mapsto \pi(\mathcal{Y})$ is a linear isomorphism; the most difficult part is to show that a \mathfrak{g} -homomorphism

$$f : (L_{\lambda_1} \otimes L_{\lambda_2})/W \rightarrow L_{\lambda_3}$$

extends to a (unique) intertwining operator of type $\left(\begin{smallmatrix} L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3}) \\ L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1}) L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2}) \end{smallmatrix} \right)$.

5 Integral intertwining operators among modules for affine Lie algebra vertex operator algebras

In this section we fix a nonnegative integral level ℓ . We will classify integral intertwining operators among $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules, using integral forms which were first introduced in a Lie-algebraic setting in [G]. We shall now recall these integral forms; see [M2] for more details.

The universal enveloping algebra $U(\widehat{\mathfrak{g}})$ of $\widehat{\mathfrak{g}}$ has an integral form $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$: it is the subring of $U(\widehat{\mathfrak{g}})$ generated by the vectors $(x_{\alpha} \otimes t^n)^k / k!$ for $k \geq 0$, where $n \in \mathbb{Z}$ and x_{α} is a root vector in a Chevalley basis of \mathfrak{g} ([G], see also [Mi] and [P]). We describe a basis of $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$ given in [Mi]. Consider a Chevalley basis $\{x_{\alpha}, h_i\}$ of \mathfrak{g} , where x_{α} is a root vector corresponding to the root α and h_i is the coroot corresponding to a simple root α_i ; suppose that θ is the highest root of \mathfrak{g} with respect to the simple roots $\{\alpha_i\}$. Then $\widehat{\mathfrak{g}}$ has an integral basis consisting of the vectors

$$x_{\alpha} \otimes t^n, \quad h_i \otimes t^n, \quad -h_{\theta} \otimes t^0 + \mathbf{k} \quad (5.1)$$

where $n \in \mathbb{Z}$. Given any order of this basis, a basis for $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$ consists of ordered products of elements of the following forms:

$$\frac{(x_{\alpha} \otimes t^n)^m}{m!}, \quad \binom{h_i \otimes t^0 + m - 1}{m}, \quad \binom{-h_{\theta} \otimes t^0 + \mathbf{k} + m - 1}{m} \quad (5.2)$$

where $n \in \mathbb{Z}$ and $m \geq 0$, as well as coefficients of powers of x in series of the form

$$\exp \left(\sum_{j \geq 1} (h_i \otimes t^{nj}) \frac{x^j}{j} \right) \quad (5.3)$$

for $n \in \mathbb{Z} \setminus \{0\}$.

Suppose L_{λ} is a finite-dimensional irreducible \mathfrak{g} -module where λ is a dominant integral weight of \mathfrak{g} , and suppose v_{λ} is a highest weight vector of L_{λ} . Then $V_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda})$ and $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda})$ have integral forms $V_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda})_{\mathbb{Z}}$ and $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda})_{\mathbb{Z}}$ given by $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}) \cdot v_{\lambda}$. Given that L_{λ} is included in $V_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda})$ and $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda})$ as their lowest conformal weight spaces (recall (4.1)), we can define

$$(L_{\lambda})_{\mathbb{Z}} = L_{\lambda} \cap U_{\mathbb{Z}}(\widehat{\mathfrak{g}}) \cdot v_{\lambda},$$

a sublattice of both $V_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda})_{\mathbb{Z}}$ and $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda})_{\mathbb{Z}}$ which, by (5.2), agrees with the classical integral form of a \mathfrak{g} -module constructed in, for example, [Hu] Chapter 7. In light of the basis for $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$ described above, we have

Proposition 5.1. *The integral forms $V_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda})_{\mathbb{Z}}$ and $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda})_{\mathbb{Z}}$ are spanned by vectors of the form $P \cdot u$, where $u \in (L_{\lambda})_{\mathbb{Z}}$ and P is a product of operators of the form $x_{\alpha}(-n)^m / m!$, for α a root of \mathfrak{g} and $n, m > 0$, and coefficients of powers of x in series of the form*

$$\exp \left(\sum_{j \geq 1} h_i(-nj) \frac{x^j}{j} \right)$$

for $n > 0$.

Proof. This follows immediately from (5.2), (5.3), and an appropriate choice of order on the basis (5.1). \square

The Lie-algebraic integral forms $V_{\hat{\mathfrak{g}}}(\ell, L_{\lambda})_{\mathbb{Z}}$ and $L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda})_{\mathbb{Z}}$ are also vertex algebraic integral forms, as shown by the following two results proved in [M2]:

Theorem 5.2. *The integral form $V_{\hat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ is the vertex subring of $V_{\hat{\mathfrak{g}}}(\ell, 0)$ generated by the vectors $\frac{x_{\alpha}(-1)^k}{k!}\mathbf{1}$ where $k \geq 0$ and x_{α} is the root vector corresponding to the root α in the chosen Chevalley basis of \mathfrak{g} . Moreover, if L_{λ} is a finite-dimensional \mathfrak{g} -module with highest weight vector v_{λ} and W is $V_{\hat{\mathfrak{g}}}(\ell, L_{\lambda})$ or $L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda})$, then $W_{\mathbb{Z}}$ is the $V_{\hat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ -module generated by v_{λ} .*

Corollary 5.3. *The integral form $L_{\hat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ of $L_{\hat{\mathfrak{g}}}(\ell, 0)$ is the integral form of $L_{\hat{\mathfrak{g}}}(\ell, 0)$ as a vertex algebra generated by the vectors $\frac{x_{\alpha}(-1)^k}{k!}\mathbf{1}$ where α is a root and $k \geq 0$.*

It was also shown in [M2] that the graded \mathbb{Z} -dual of a $V_{\hat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ - or $L_{\hat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ -module is also a $V_{\hat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ - or $L_{\hat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ -module. In particular, suppose that λ is a dominant integral weight of \mathfrak{g} satisfying $\langle \lambda, \theta \rangle \leq \ell$ where θ is the highest root of \mathfrak{g} , so that $L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda})$ is an $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -module. Then the graded \mathbb{Z} -dual of $L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda})_{\mathbb{Z}}$ is the $L_{\hat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ -module $L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda})'_{\mathbb{Z}}$, which is an integral form of $L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda}^*)$ that is generally different from $L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda}^*)_{\mathbb{Z}}$.

Remark 5.4. The precise identity of the integral form $L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda})'_{\mathbb{Z}} \subseteq L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda}^*)$ depends on the choice of isomorphism $L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda})' \cong L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda}^*)$. Because $L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda})$ is irreducible, such an isomorphism amounts to a \mathfrak{g} -module isomorphism $L_{\lambda}^* \cong L_{\lambda}$, in which the \mathbb{Z} -dual of $(L_{\lambda})_{\mathbb{Z}}$ is identified with a lattice $(L_{\lambda})'_{\mathbb{Z}} \subseteq L_{\lambda}^*$.

We now come to our main theorem on integral intertwining operators among $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules. We consider intertwining operators among three $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules $L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda_i})$ for $i = 1, 2, 3$. In the statement of the following theorem, we use $U_{\mathbb{Z}}(\mathfrak{g})$ to denote $U_{\mathbb{Z}}(\hat{\mathfrak{g}}) \cap U(\mathfrak{g})$, which by (5.2) has as a basis ordered monomials in elements of the forms

$$\frac{x_{\alpha}^m}{m!}, \quad \binom{h_i + m - 1}{m}$$

where $m \geq 0$ and we identify $x_{\alpha} = x_{\alpha} \otimes t^0$ and $h_i = h_i \otimes t^0$.

Theorem 5.5. *The lattice of intertwining operators within $V_{L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda_1}) L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda_2})}^{L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda_3})}$ which are integral with respect to $L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda_1})_{\mathbb{Z}}$, $L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda_2})_{\mathbb{Z}}$, and $L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda_3}^*)'_{\mathbb{Z}}$ is isomorphic to*

$$\text{Hom}_{U_{\mathbb{Z}}(\mathfrak{g})}(((L_{\lambda_1})_{\mathbb{Z}} \otimes_{\mathbb{Z}} (L_{\lambda_2})_{\mathbb{Z}})/W_{\mathbb{Z}}, (L_{\lambda_3}^*)'_{\mathbb{Z}})$$

where $W_{\mathbb{Z}}$ is the $U_{\mathbb{Z}}(\mathfrak{g})$ -submodule of $(L_{\lambda_1})_{\mathbb{Z}} \otimes_{\mathbb{Z}} (L_{\lambda_2})_{\mathbb{Z}}$ generated by vectors of the form

$$v_{\lambda_1} \otimes \frac{x_{\theta}^{\ell - \langle \lambda_1, \theta \rangle + 1}}{(\ell - \langle \lambda_1, \theta \rangle + 1)!} \cdot w.$$

Here v_{λ_1} is a highest weight vector generating $(L_{\lambda_1})_{\mathbb{Z}}$ as a $U_{\mathbb{Z}}(\mathfrak{g})$ -module, and $w \in (L_{\lambda_2})_{\mathbb{Z}}$.

Remark 5.6. Note that $(L_{\lambda_3^*})'_{\mathbb{Z}}$ is an integral form of L_{λ_3} , the lowest conformal weight space of $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})$.

Remark 5.7. We use vectors of the form $v_{\lambda_1} \otimes \frac{x_{\theta}^{\ell - \langle \lambda_1, \theta \rangle + 1}}{(\ell - \langle \lambda_1, \theta \rangle + 1)!} \cdot w$ to generate $W_{\mathbb{Z}}$, rather than vectors of the form $v_{\lambda_1} \otimes x_{\theta}^{\ell - \langle \lambda_1, \theta \rangle + 1} \cdot w$, to avoid unnecessary torsion in the quotient $((L_{\lambda_1})_{\mathbb{Z}} \otimes_{\mathbb{Z}} (L_{\lambda_2})_{\mathbb{Z}}) / W_{\mathbb{Z}}$.

Remark 5.8. Note that this theorem determines which intertwining operators of type $\binom{L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})}{L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1}) L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})}$ satisfy

$$\langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)} \rangle \in \mathbb{Z}\{x\}$$

for any $w_{(1)} \in L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})_{\mathbb{Z}}$, $w_{(2)} \in L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})_{\mathbb{Z}}$, and $w'_{(3)} \in L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3^*})_{\mathbb{Z}}$, rather than which intertwining operators satisfy

$$\mathcal{Y} : L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})_{\mathbb{Z}} \otimes L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})_{\mathbb{Z}} \rightarrow L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})_{\mathbb{Z}}\{x\}.$$

It is not clear that there will generally be any non-zero intertwining operators which satisfy the latter condition.

Proof. First suppose an intertwining operator \mathcal{Y} is integral with respect to $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})_{\mathbb{Z}}$, $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})_{\mathbb{Z}}$, and $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3^*})'_{\mathbb{Z}}$. Then the map

$$\pi(\mathcal{Y}) : w_{(1)} \otimes w_{(2)} \mapsto w_{(1)}(0)w_{(2)}$$

given in (4.3) sends $(L_{\lambda_1})_{\mathbb{Z}} \otimes_{\mathbb{Z}} (L_{\lambda_2})_{\mathbb{Z}}$ into $(L_{\lambda_3^*})'_{\mathbb{Z}}$ and equals zero on $W_{\mathbb{Z}} \subseteq W = \langle v_{\lambda_1} \otimes x_{\theta}^{\ell - \langle \lambda_1, \theta \rangle + 1} \cdot w \rangle$.

Conversely, suppose

$$f : (L_{\lambda_1})_{\mathbb{Z}} \otimes_{\mathbb{Z}} (L_{\lambda_2})_{\mathbb{Z}} \rightarrow (L_{\lambda_3^*})'_{\mathbb{Z}}$$

is a $U_{\mathbb{Z}}(\mathfrak{g})$ -homomorphism which equals zero on $W_{\mathbb{Z}}$. Then f extends to a $U(\mathfrak{g})$ -homomorphism from $L_{\lambda_1} \otimes L_{\lambda_2}$ to L_{λ_3} which equals zero on W . Thus since π given in (4.3) is an isomorphism, f induces a unique intertwining operator

$$\mathcal{Y}_f : L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1}) \otimes L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2}) \rightarrow L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})\{x\}$$

which satisfies $\pi(\mathcal{Y}_f) = f$ and

$$\langle w'_{(3)}, \mathcal{Y}_f(w_{(1)}, x)w_{(2)} \rangle \in \mathbb{Z}\{x\} \tag{5.4}$$

for $w_{(1)} \in (L_{\lambda_1})_{\mathbb{Z}}$, $w_{(2)} \in (L_{\lambda_2})_{\mathbb{Z}}$, and $w'_{(3)} \in (L_{\lambda_3^*})_{\mathbb{Z}}$. To complete the proof, it is enough to show that (5.4) holds for all $w_{(1)} \in L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})_{\mathbb{Z}}$, $w_{(2)} \in L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})_{\mathbb{Z}}$, and $w'_{(3)} \in L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3^*})_{\mathbb{Z}}$.

Since $(L_{\lambda_1})_{\mathbb{Z}}$ and $(L_{\lambda_2})_{\mathbb{Z}}$ generate $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})_{\mathbb{Z}}$ and $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})_{\mathbb{Z}}$, respectively, as $L_{\widehat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ -modules, by Theorem 3.9 it is enough to show that (5.4) holds for $w_{(1)} \in (L_{\lambda_1})_{\mathbb{Z}}$, $w_{(2)} \in (L_{\lambda_2})_{\mathbb{Z}}$, and $w'_{(3)} \in L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3^*})_{\mathbb{Z}}$. Since (5.4) holds when $w'_{(3)} \in (L_{\lambda_3^*})'_{\mathbb{Z}}$, by Proposition 5.1 it is enough to show that if it holds for some particular $w'_{(3)} \in L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3^*})_{\mathbb{Z}}$, then it also holds for the coefficients of powers of y in

$$\exp(x_{\alpha}(-n)y) \cdot w'_{(3)}$$

where α is a root and $n > 0$, and for coefficients of powers of y in

$$\exp \left(\sum_{j \geq 1} h_i(-nj) \frac{y^j}{j} \right) \cdot w'_{(3)}$$

where $n > 0$.

We will need to use the following commutator formula which holds for any $a \in \mathfrak{g}$, $m \in \mathbb{Z}$, and intertwining operator \mathcal{Y} , which follows from the Jacobi identity (3.1) by setting $v = a(-1)\mathbf{1}$ and taking the coefficient of $x_0^{-1}x_1^{-m-1}$: for $w_{(1)} \in L_{\lambda_1}$,

$$[a(m), \mathcal{Y}(w_{(1)}, x)] = x^m \mathcal{Y}(a(0)w_{(1)}, x).$$

Then if also $w_{(2)} \in L_{\lambda_2}$ and $m > 0$,

$$a(m) \mathcal{Y}(w_{(1)}, x) w_{(2)} = x^m \mathcal{Y}(a(0)w_{(1)}, x) w_{(2)}. \quad (5.5)$$

We will also use the fact that for any $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -module W , $w \in W$, $w' \in W'$, $a \in \mathfrak{g}$, and $n \in \mathbb{Z}$,

$$\langle a(n)w', w \rangle = \langle w', -a(-n)w \rangle.$$

This follows from (2.1) and (4.2).

Now suppose (5.4) holds for $w'_{(3)} \in L_{\hat{\mathfrak{g}}}(\ell, L_{\lambda_3^*})_{\mathbb{Z}}$ and $n > 0$; then by (5.5),

$$\begin{aligned} \langle \exp(x_{\alpha}(-n)y) \cdot w'_{(3)}, \mathcal{Y}_f(w_{(1)}, x) w_{(2)} \rangle &= \langle w'_{(3)}, \exp(-x_{\alpha}(n)y) \mathcal{Y}_f(w_{(1)}, x) w_{(2)} \rangle \\ &= \langle w'_{(3)}, \mathcal{Y}_f(\exp(-x_{\alpha}(0)x^n y) \cdot w_{(1)}, x) w_{(2)} \rangle \in \mathbb{Z}\{x, y\} \end{aligned}$$

because for any $m \geq 0$, $\frac{x_{\alpha}(0)^m}{m!} \cdot w_{(1)} \in (L_{\lambda_1})_{\mathbb{Z}}$. Additionally, for each $n > 0$,

$$\begin{aligned} &\left\langle \exp \left(\sum_{j \geq 1} h_i(-nj) \frac{y^j}{j} \right) \cdot w'_{(3)}, \mathcal{Y}_f(w_{(1)}, x) w_{(2)} \right\rangle \\ &= \left\langle w'_{(3)}, \exp \left(\sum_{j \geq 1} -h_i(nj) \frac{y^j}{j} \right) \mathcal{Y}_f(w_{(1)}, x) w_{(2)} \right\rangle \\ &= \left\langle w'_{(3)}, \mathcal{Y}_f \left(\exp \left(-h_i(0) \sum_{j \geq 1} (-1)^j \frac{(-x^n y)^j}{j} \right) \cdot w_{(1)}, x \right) w_{(2)} \right\rangle \\ &= \langle w'_{(3)}, \mathcal{Y}_f(\exp(h_i(0) \log(1 - x^n y)) \cdot w_{(1)}, x) w_{(2)} \rangle \\ &= \langle w'_{(3)}, \mathcal{Y}_f((1 - x^n y)^{h_i(0)} \cdot w_{(1)}, x) w_{(2)} \rangle \in \mathbb{Z}\{x, y\} \end{aligned}$$

because

$$(1 - x^n y)^{h_i(0)} = \sum_{k \geq 0} (-1)^k \binom{h_i(0)}{k} (x^n y)^k.$$

Note that $\binom{h_i(0)}{k} \cdot w_{(1)} \in (L_{\lambda_1})_{\mathbb{Z}}$ because λ_1 is a dominant integral weight of \mathfrak{g} , and thus $h_i(0)$ acts on basis elements of $(L_{\lambda_1})_{\mathbb{Z}}$ as integers. This completes the proof. \square

6 Lattice vertex operator algebras, their modules, and intertwining operators

In this section we recall the construction of vertex operator algebras from even lattices, their modules, and intertwining operators among their modules; see [FLM], [DL], and [LL] for more details. We fix a nondegenerate even lattice L with symmetric bilinear form $\langle \cdot, \cdot \rangle$, and then set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$, an abelian Lie algebra. Thus we can form the Heisenberg vertex operator algebra $M(1) = V_{\mathfrak{h}}(1, 0)$.

We also need the twisted group algebra of L° , the dual lattice of L , which is constructed from a central extension of L° by a finite cyclic group. For our purposes here, we may view the twisted group algebra $\mathbb{C}\{L^\circ\}$ as the associative algebra with basis vectors e_β for $\beta \in L^\circ$ with identity e_0 and product

$$e_\beta e_\gamma = \varepsilon(\beta, \gamma) e_{\beta+\gamma}.$$

Here ε maps each pair of lattice elements to a root of unity and may be chosen to satisfy

$$\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma) \varepsilon(\beta, \gamma), \quad \varepsilon(\alpha, \beta + \gamma) = \varepsilon(\alpha, \beta) \varepsilon(\alpha, \gamma)$$

for all $\alpha, \beta, \gamma \in L^\circ$. We will also use the commutator map c of our central extension, which is defined by

$$e_\beta e_\gamma = c(\beta, \gamma) e_\gamma e_\beta$$

for $\beta, \gamma \in L^\circ$. Note that from the definitions,

$$c(\beta, \gamma) = \varepsilon(\beta, \gamma) \varepsilon(\gamma, \beta)^{-1} \tag{6.1}$$

for all $\beta, \gamma \in L^\circ$. The commutator map c is required to satisfy

$$c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$$

for $\alpha, \beta \in L$. (Note that $L \subseteq L^\circ$ since L is even, and thus integral.)

We define an action of $\mathfrak{h} = \mathfrak{h} \otimes t^0$ on $\mathbb{C}\{L^\circ\}$ by

$$h(0)e_\beta = \langle h, \beta \rangle e_\beta$$

for $h \in \mathfrak{h}$, $\beta \in L^\circ$. Also for $\beta \in L^\circ$ and x a formal variable, we define a map x^β on $\mathbb{C}\{L^\circ\}$ by

$$x^\beta e_\gamma = e_\gamma x^{\langle \beta, \gamma \rangle}$$

for any $\gamma \in L^\circ$.

For any subset $S \subseteq L^\circ$, we use $\mathbb{C}\{S\}$ to denote the subspace of $\mathbb{C}\{L^\circ\}$ spanned by the vectors e_β for $\beta \in S$. Then as a vector space, the vertex operator algebra associated to L is

$$V_L = M(1) \otimes \mathbb{C}\{L\}.$$

For $\alpha \in L$, we use the notation $\iota(e_\alpha)$ to denote the element $1 \otimes e_\alpha \in V_L$. The vacuum of V_L is $\mathbf{1} = \iota(e_0)$, and the vertex algebra structure is generated by the vectors $\iota(e_\alpha)$ for $\alpha \in L$, whose vertex operators are

$$Y(\iota(e_\alpha), x) = E^-(-\alpha, x) E^+(-\alpha, x) e_\alpha x^\alpha \tag{6.2}$$

where

$$E^\pm(\beta, x) = \exp \left(\sum_{n \in \pm \mathbb{Z}_+} \frac{\beta(n)}{n} x^{-n} \right)$$

for any $\beta \in L^\circ$, and e_α denotes the left multiplication action on $\mathbb{C}\{L\}$.

Remark 6.1. The vertex algebra V_L is a vertex operator algebra in the sense of [FLM] only when L is positive definite. If L is not positive definite, V_L is a conformal vertex algebra in the sense of [HLZ1]: there are infinitely many weight spaces of negative conformal weight, and the weight spaces are infinite dimensional. However, all the results of this paper still hold in this generality.

We now recall the irreducible modules for V_L . The space

$$V_{L^\circ} = M(1) \otimes \mathbb{C}\{L^\circ\}$$

is a V_L -module, with vertex operators for the $\iota(e_\alpha)$ acting on V_{L° the same as in (6.2). Moreover, the submodules

$$V_{\beta+L} = M(1) \otimes \mathbb{C}\{\beta + L\},$$

where β runs over coset representatives of L°/L , exhaust the irreducible V_L -modules up to equivalence.

We now recall the intertwining operators among V_L -modules from [DL]. For any $\beta \in L^\circ$, we have an intertwining operator

$$\mathcal{Y}_\beta : V_{\beta+L} \otimes V_{L^\circ} \rightarrow V_{L^\circ}\{x\},$$

where for any $\gamma \in \beta + L$,

$$\mathcal{Y}_\beta(\iota(e_\gamma), x) = E^-(-\gamma, x) E^+(-\gamma, x) e_\gamma x^\gamma e^{\pi i \beta} c(\cdot, \beta). \quad (6.3)$$

Here, the operator $e^{\pi i \beta}$ on V_{L° is given by

$$e^{\pi i \beta} \cdot u \otimes \iota(e_\gamma) = e^{\pi i \langle \beta, \gamma \rangle} u \otimes \iota(e_\gamma)$$

for $u \in M(1)$, $\gamma \in L^\circ$, and the operator $c(\cdot, \beta)$ is defined by

$$c(\cdot, \beta) \cdot u \otimes \iota(e_\gamma) = c(\gamma, \beta) u \otimes \iota(e_\gamma).$$

Then for any $\gamma \in L^\circ$, $\mathcal{Y}_\beta|_{V_{\beta+L} \otimes V_{\gamma+L}}$ is an intertwining operator of type $\begin{pmatrix} V_{\beta+\gamma+L} \\ V_{\beta+L} V_{\gamma+L} \end{pmatrix}$. From [DL] we have:

Proposition 6.2. *For any $\alpha, \beta, \gamma \in L^\circ$,*

$$V_{\beta\gamma}^\alpha = V_{V_{\beta+L} V_{\gamma+L}}^{V_{\alpha+L}} = \begin{cases} \mathbb{C} \mathcal{Y}_\beta|_{V_{\beta+L} \otimes V_{\gamma+L}} & \text{if } \alpha = \beta + \gamma \\ 0 & \text{if } \alpha \neq \beta + \gamma \end{cases}.$$

Remark 6.3. Note that since for any $\beta \in L^\circ$, $\mathcal{Y}_\beta|_{V_{\beta+L} \otimes V_{-\beta+L}}$ is a non-zero intertwining operator of type $\begin{pmatrix} V_L \\ V_{\beta+L} V_{-\beta+L} \end{pmatrix}$, Proposition 3.12 implies that $V_{\beta+L}' \cong V_{-\beta+L}$.

7 Integral intertwining operators among modules for lattice vertex operator algebras

We continue to fix a nondegenerate even lattice L in this section. We will classify the integral intertwining operators among V_L -modules, using the integral forms for V_L -modules that were introduced in [M2]. (The integral form of V_L itself was first introduced in [B].) We recall these integral forms by stating the following theorem from [M2] (the algebra part was also proved in [P] and [DG]):

Theorem 7.1. *The vertex subring $V_{L,\mathbb{Z}}$ of V_L generated by the vectors $\iota(e_\alpha)$ for $\alpha \in L$ is an integral form of V_L . Moreover, for any $\beta \in L^\circ$, the $V_{L,\mathbb{Z}}$ -submodule $V_{\beta+L,\mathbb{Z}}$ of $V_{\beta+L}$ generated by $\iota(e_\beta)$ is an integral form of $V_{\beta+L}$.*

Remark 7.2. For this theorem to hold, we must assume that ε satisfies $\varepsilon(\alpha, \beta) = \pm 1$ for any $\alpha, \beta \in L$. In fact the proof (although not the statement) of Lemma 4.1 in [M2] shows that we can choose ε to satisfy the stronger condition $\varepsilon(\alpha, \beta) = \pm 1$ whenever either $\alpha \in L$ or $\beta \in L$.

Remark 7.3. The proof of Theorem 7.1 that is given in [M2], using the weaker condition in Remark 7.2, shows that for $\beta \in L^\circ$, $V_{\beta+L,\mathbb{Z}}$ is the \mathbb{Z} -span of coefficients of products of the form

$$E^-(-\alpha_1, x_1) \cdots E^-(-\alpha_k, x_k) \iota(e_\alpha e_\beta)$$

where $\alpha_1, \dots, \alpha_k, \alpha \in L$. If we assume the stronger condition in Remark 7.2, then for any $\beta \in L^\circ$, $V_{\beta+L,\mathbb{Z}}$ is the \mathbb{Z} -span of coefficients of products of the form

$$E^-(-\alpha_1, x_1) \cdots E^-(-\alpha_k, x_k) \iota(e_\gamma)$$

where $\alpha_1, \dots, \alpha_k \in L$ and $\gamma \in \beta + L$. Note that the proof of Theorem 7.1 which is given in [M2] incorrectly states that the second description of $V_{\beta+L,\mathbb{Z}}$ follows from the weaker condition in Remark 7.2. In the rest of this section, we will generally use the first description of $V_{\beta+L,\mathbb{Z}}$ since some of the formulas we need are slightly simpler this way.

Remark 7.4. The integral form $V_{\beta+L,\mathbb{Z}}$ depends on the choice of ε , but once an ε satisfying the stronger condition of Remark 7.2 is fixed, the second description of $V_{\beta+L,\mathbb{Z}}$ in Remark 7.3 shows that it does not depend on the choice of generator. That is, the $V_{L,\mathbb{Z}}$ -module in $V_{\beta+L}$ generated by $\iota(e_\beta)$ is the same as the $V_{L,\mathbb{Z}}$ -module generated by $\iota(e_{\beta'})$ for any $\beta' \in \beta + L$.

In order to obtain integral intertwining operators, we will need an explicit description, as in Remark 7.3, of the graded \mathbb{Z} -dual of the integral form of a V_L -module. Recalling that the contragredient of $V_{\beta+L}$ is $V_{-\beta+L}$, we shall use an invariant bilinear pairing as in (3.10) to identify $V'_{-\beta+L,\mathbb{Z}}$ as a sublattice of $V_{\beta+L,\mathbb{Z}}$. First we calculate (3.10) with $\mathcal{Y} = \mathcal{Y}_\beta$ and with the form $(\cdot, \cdot)_{V_L}$ on V_L normalized so that $(\mathbf{1}, \mathbf{1})_{V_L} = 1$. Thus for $u \in V_{\beta+L}$ and $v \in V_{-\beta+L}$, we have

$$(u, v) = \text{Res}_x (\mathbf{1}, (\mathcal{Y}_\beta)_0^o(u, e^{\pi i} x) e^{xL(1)} v)_{V_L}.$$

In particular, (6.3) implies that for $\gamma \in \beta + L$, $\gamma' \in -\beta + L$,

$$\begin{aligned}
& (\iota(e_\gamma), \iota(e_{\gamma'})) \\
&= \text{Res}_x x^{-1} (\mathbf{1}, e^{-\pi i \langle \gamma, \gamma \rangle / 2} x^{-\langle \gamma, \gamma \rangle} E^-(-\gamma, -x^{-1}) e_\gamma (e^{\pi i} x)^{-\langle \gamma, \gamma' \rangle} e^{\pi i \langle \beta, \gamma' \rangle} c(\gamma', \beta) \iota(e_{\gamma'}))_{V_L} \\
&= \text{Res}_x x^{-1} (\mathbf{1}, e^{\pi i (\langle \beta - \gamma, \gamma' \rangle - \langle \gamma, \gamma \rangle / 2)} x^{-\langle \gamma, \gamma + \gamma' \rangle} E^-(-\gamma, -x^{-1}) c(\gamma', \beta) \varepsilon(\gamma, \gamma') \iota(e_{\gamma + \gamma'}))_{V_L} \\
&= e^{\pi i \langle \gamma - 2\beta, \gamma \rangle / 2} c(\gamma, \beta)^{-1} \varepsilon(\gamma, \gamma)^{-1} \delta_{\gamma + \gamma', 0}.
\end{aligned} \tag{7.1}$$

Using (7.1), we see that for any $\alpha \in L$,

$$\begin{aligned}
(\iota(e_\alpha e_\beta), \iota(e_{-\alpha} e_{-\beta})) &= \varepsilon(\alpha, \beta) \varepsilon(-\alpha, -\beta) (\iota(e_{\alpha + \beta}), \iota(e_{-\alpha - \beta})) \\
&= \varepsilon(\alpha, \beta)^2 e^{\pi i \langle \alpha - \beta, \alpha + \beta \rangle / 2} c(\alpha + \beta, \beta)^{-1} \varepsilon(\alpha + \beta, \alpha + \beta)^{-1} \\
&= e^{\pi i \langle \alpha, \alpha \rangle / 2} \varepsilon(\alpha, \alpha)^{-1} e^{-\pi i \langle \beta, \beta \rangle / 2} c(\beta, \beta)^{-1} \varepsilon(\beta, \beta)^{-1},
\end{aligned}$$

since $c(\alpha, \beta) = \varepsilon(\alpha, \beta) \varepsilon(\beta, \alpha)^{-1}$ from (6.1).

If we now renormalize the invariant pairing (\cdot, \cdot) between $V_{\beta + L}$ and $V_{-\beta + L}$ by setting

$$(\cdot, \cdot)_{\text{new}} = e^{\pi i \langle \beta, \beta \rangle / 2} c(\beta, \beta) \varepsilon(\beta, \beta) (\cdot, \cdot)_{\text{old}},$$

we see that now

$$(\iota(e_\alpha e_\beta), \iota(e_{\alpha'} e_{-\beta})) = (-1)^{\langle \alpha, \alpha \rangle / 2} \varepsilon(\alpha, \alpha)^{-1} \delta_{\alpha + \alpha', 0} = 0 \text{ or } \pm 1$$

for any $\alpha, \alpha' \in L$, using Remark 7.2. We can use this new invariant bilinear pairing between $V_{\beta + L}$ and $V_{-\beta + L}$, which is the form in (3.10) with

$$\mathcal{Y} = e^{\pi i \langle \beta, \beta \rangle / 2} c(\beta, \beta) \varepsilon(\beta, \beta) \mathcal{Y}_\beta|_{V_{\beta + L} \otimes V_{-\beta + L}},$$

to identify $V'_{-\beta + L, \mathbb{Z}}$ as a sublattice of $V_{\beta + L, \mathbb{Z}}$, for any $\beta \in L^\circ$:

Proposition 7.5. *For any $\beta \in L^\circ$, the integral form $V'_{-\beta + L, \mathbb{Z}}$ is the integral form of $V_{\beta + L}$ integrally spanned by coefficients of products of the form*

$$E^-(-\beta_1, x_1) \cdots E^-(-\beta_k, x_k) \iota(e_\alpha e_\beta) \tag{7.2}$$

where $\beta_i \in L^\circ$ and $\alpha \in L$.

Proof. Let $\tilde{V}_{\beta + L, \mathbb{Z}}$ be the integral form of $V_{\beta + L}$ integrally spanned by coefficients of products as in (7.2). We first show that $\tilde{V}_{\beta + L, \mathbb{Z}} \subseteq V'_{-\beta + L, \mathbb{Z}}$. For $\beta \in L^\circ$ and $n \in \mathbb{Z}$, it is easy to see from (2.1) that the adjoint of the operator $\beta(n)$ is $-\beta(-n)$. Thus the adjoint of

$$E^-(-\beta, x) = \exp \left(\sum_{n < 0} \frac{-\beta(n)}{n} x^{-n} \right)$$

is

$$\exp \left(\sum_{n < 0} \frac{\beta(-n)}{n} x^{-n} \right) = \exp \left(\sum_{n > 0} \frac{-\beta(n)}{n} x^n \right) = E^+(-\beta, x^{-1}).$$

From Proposition 4.3.1 in [FLM], we have

$$E^+(\beta, x_1)E^-(\gamma, x_2) = E^-(\gamma, x_2)E^+(\beta, x_1) \left(1 - \frac{x_2}{x_1}\right)^{\langle \beta, \gamma \rangle}$$

for any $\beta, \gamma \in L^\circ$. Thus for any $\beta_1, \dots, \beta_k \in L^\circ$ and $\alpha_1, \dots, \alpha_{k'}, \alpha, \alpha' \in L$,

$$\begin{aligned} & (E^-(\beta_1, x_1) \cdots E^-(\beta_k, x_k) \iota(e_\alpha e_\beta), E^-(\alpha_1, y_1) \cdots E^-(\alpha_{k'}, y_{k'}) \iota(e_{\alpha'} e_{-\beta})) \\ &= (\iota(e_\alpha e_\beta), E^+(-\beta_1, x_1^{-1}) \cdots E^+(-\beta_k, x_k^{-1}) E^-(\alpha_1, y_1) \cdots E^-(\alpha_{k'}, y_{k'}), \iota(e_{\alpha'} e_{-\beta})) \\ &= (\iota(e_\alpha e_\beta), E^-(\alpha_1, y_1) \cdots E^-(\alpha_{k'}, y_{k'}) \iota(e_{\alpha'} e_{-\beta})) \prod_{i=1}^k \prod_{j=1}^{k'} (1 - x_i y_j)^{\langle \beta_i, \alpha_j \rangle} \\ &= (-1)^{\langle \alpha, \alpha \rangle / 2} \varepsilon(\alpha, \alpha)^{-1} \delta_{\alpha+\alpha', 0} \prod_{i=1}^k \prod_{j=1}^{k'} (1 - x_i y_j)^{\langle \beta_i, \alpha_j \rangle} \in \mathbb{Z}[[\{x_i\}, \{y_j\}]] \end{aligned} \quad (7.3)$$

since each $\langle \beta_i, \alpha_j \rangle \in \mathbb{Z}$. Since $V_{-\beta+L, \mathbb{Z}}$ is the integral span of coefficients of products of the form

$$E^-(\alpha_1, y_1) \cdots E^-(\alpha_{k'}, y_{k'}) \iota(e_{\alpha'} e_{-\beta})$$

for $\alpha_1, \dots, \alpha_{k'}, \alpha' \in L$ by Remark 7.3, it follows that $\tilde{V}_{\beta+L, \mathbb{Z}} \subseteq V'_{-\beta+L, \mathbb{Z}}$.

Now we must show the opposite inclusion $V'_{-\beta+L, \mathbb{Z}} \subseteq \tilde{V}_{\beta+L, \mathbb{Z}}$. For any $\beta \in L^\circ$ and $n > 0$, let $s_{\beta, n}$ denote the coefficient of x^n in $E^-(\beta, x)$, and for any partition $\Lambda = (\lambda_1, \dots, \lambda_k)$ where $\lambda_1 \geq \dots \geq \lambda_k > 0$ we define

$$h_\Lambda(\beta) = s_{\beta, \lambda_1} \cdots s_{\beta, \lambda_k}.$$

If $\{\alpha^{(1)}, \dots, \alpha^{(l)}\}$ is a basis for L , then $V_{-\beta+L, \mathbb{Z}}$ has a basis consisting of the vectors

$$h_{\Lambda_1}(\alpha^{(1)}) \cdots h_{\Lambda_l}(\alpha^{(l)}) \iota(e_{-\alpha} e_{-\beta}) \quad (7.4)$$

where $\alpha \in L$ and $\Lambda_1, \dots, \Lambda_k$ run over all partitions (see [M1] and [DG]).

Let $\{\beta^{(1)}, \dots, \beta^{(l)}\}$ be the basis of L° dual to $\{\alpha^{(1)}, \dots, \alpha^{(l)}\}$. Then in the basis of $V'_{-\beta+L, \mathbb{Z}}$ dual to (7.4), the basis vector corresponding to $h_\Lambda(\alpha^{(i)}) \iota(e_{-\alpha} e_{-\beta})$ for any i , $\alpha \in L$, and partition Λ is given by

$$(-1)^{\langle \alpha, \alpha \rangle / 2} \varepsilon(\alpha, \alpha) m_\Lambda(\alpha^{(i)}) \iota(e_\alpha e_\beta)$$

where $m_\Lambda(\alpha^{(i)})$ is a polynomial in the operators $\beta^{(i)}(-n)$, $n > 0$. It follows that the basis vector dual to the vector (7.4) is

$$(-1)^{\langle \alpha, \alpha \rangle / 2} \varepsilon(\alpha, \alpha) m_{\Lambda_1}(\alpha^{(1)}) \cdots m_{\Lambda_k}(\alpha^{(l)}) \iota(e_\alpha e_\beta).$$

Thus it is enough to show that for any i and partition Λ , $m_\Lambda(\alpha^{(i)})$ is an integer linear combination of operators $h_{\Lambda'}(-\beta^{(i)})$ for partitions Λ' .

We will need to use the well-known monomial and complete symmetric polynomials (see for example [Mac] Chapter 1, Section 2). Fix a positive integer n . Then for any partition $\Lambda = (\lambda_1, \dots, \lambda_k)$ where $k \leq n$, the monomial symmetric polynomial in n variables $m_\Lambda(y_1, \dots, y_n)$ is the sum of all distinct permutations of the monomial $y_1^{\lambda_1} \cdots y_k^{\lambda_k}$. The

polynomials $m_\Lambda(y_1, \dots, y_n)$ as Λ runs over partitions with less than or equal to n parts form a basis of the (graded) ring of n -variable symmetric polynomials with integer coefficients. On the other hand, for any $m > 0$, the complete homogeneous symmetric polynomial $h_m(y_1, \dots, y_n)$ is the sum of all distinct degree m monomials in n variables; for any partition $\Lambda = (\lambda_1, \dots, \lambda_k)$, we define the polynomial

$$h_\Lambda(y_1, \dots, y_n) = h_{\lambda_1}(y_1, \dots, y_n) \cdots h_{\lambda_k}(y_1, \dots, y_n).$$

The polynomials $h_\Lambda(y_1, \dots, y_n)$ as Λ runs over all partitions with parts less than or equal to n forms another basis for the ring of n -variable symmetric polynomials. In particular, for any partition Λ of n ,

$$m_\Lambda(y_1, \dots, y_n) = \sum k_{\Lambda'} h_{\Lambda'}(y_1, \dots, y_n) \quad (7.5)$$

where $k_{\Lambda'} \in \mathbb{Z}$ and Λ' runs over all partitions of n .

In light of (7.5), we will show that for any partition $\Lambda = (\lambda_1, \dots, \lambda_k)$ of an integer n , any i , and $\alpha \in L$,

$$\begin{aligned} & (m_\Lambda(\alpha^{(i)})\iota(e_\alpha e_\beta), E^-(-\alpha^{(i)}, y_1) \cdots E^-(-\alpha^{(i)}, y_n) \iota(e_{-\alpha} e_{-\beta})) \\ &= (-1)^{\langle \alpha, \alpha \rangle / 2} \varepsilon(\alpha, \alpha)^{-1} m_\Lambda(y_1, \dots, y_n) \end{aligned} \quad (7.6)$$

and

$$\begin{aligned} & (h_\Lambda(-\beta^{(i)})\iota(e_\alpha e_\beta), E^-(-\alpha^{(i)}, y_1) \cdots E^-(-\alpha^{(i)}, y_n) \iota(e_{-\alpha} e_{-\beta})) \\ &= (-1)^{\langle \alpha, \alpha \rangle / 2} \varepsilon(\alpha, \alpha)^{-1} h_\Lambda(y_1, \dots, y_n). \end{aligned} \quad (7.7)$$

Since $m_\Lambda(\alpha^{(i)})$ and $h_\Lambda(-\beta^{(i)})$ are operators of degree n and are polynomials in the operators $\beta^{(i)}(-m)$ for $m > 0$, $V_{-\beta+L, \mathbb{Z}}$ is spanned by coefficients of monomials in

$$E^-(-\alpha^{(i)}, y_1) \cdots E^-(-\alpha^{(i)}, y_n) \iota(e_{-\alpha} e_{-\beta}) \quad (7.8)$$

and by vectors v such that

$$(m_\Lambda(\alpha^{(i)})\iota(e_\alpha e_\beta), v) = (h_\Lambda(-\beta^{(i)})\iota(e_\alpha e_\beta), v) = 0.$$

Thus (7.5), (7.6), and (7.7) imply that

$$m_\Lambda(\alpha^{(i)})\iota(e_\alpha e_\beta) = \sum k_{\Lambda'} h_{\Lambda'}(-\beta^{(i)})\iota(e_\alpha e_\beta),$$

and thus

$$m_\Lambda(\alpha^{(i)}) = \sum k_{\Lambda'} h_{\Lambda'}(-\beta^{(i)}),$$

which will complete the proof.

To verify (7.6), we observe that by the definition of $m_\Lambda(\alpha^{(i)})$, a monomial on the left side of (7.6) has non-zero coefficient (equal to $(-1)^{\langle \alpha, \alpha \rangle / 2} \varepsilon(\alpha, \alpha)^{-1}$) if and only if the corresponding coefficient of (7.8) is $h_\Lambda(\alpha^{(i)})\iota(e_{-\alpha} e_{-\beta})$. Since the coefficient of a monomial in (7.8) equals $h_\Lambda(\alpha^{(i)})\iota(e_{-\alpha} e_{-\beta})$ if and only if the monomial is a permutation of $y_1^{\lambda_1} \cdots y_k^{\lambda_k}$, (7.6) follows from the definition of $m_\Lambda(y_1, \dots, y_n)$.

To verify (7.7), we see from (7.3) that the left side of (7.7) is the coefficient of $x_1^{\lambda_1} \cdots x_k^{\lambda_k}$ in

$$\begin{aligned}
& (-1)^{\langle \alpha, \alpha \rangle / 2} \varepsilon(\alpha, \alpha)^{-1} \prod_{i=1}^k \prod_{j=1}^n (1 - x_i y_j)^{-1} = (-1)^{\langle \alpha, \alpha \rangle / 2} \varepsilon(\alpha, \alpha)^{-1} \prod_{i=1}^k \prod_{j=1}^n \sum_{m \geq 0} x_i^m y_j^m \\
& = (-1)^{\langle \alpha, \alpha \rangle / 2} \varepsilon(\alpha, \alpha)^{-1} \prod_{i=1}^k \sum_{m \geq 0} \left(\sum_{\substack{m_1, \dots, m_n \geq 0 \\ m_1 + \dots + m_n = m}} y_1^{m_1} \cdots y_n^{m_n} \right) x_i^m \\
& = (-1)^{\langle \alpha, \alpha \rangle / 2} \varepsilon(\alpha, \alpha)^{-1} \prod_{i=1}^k \sum_{m \geq 0} h_m(y_1, \dots, y_n) x_i^m.
\end{aligned}$$

Thus the coefficient of $x_1^{\lambda_1} \cdots x_k^{\lambda_k}$ is

$$(-1)^{\langle \alpha, \alpha \rangle / 2} \varepsilon(\alpha, \alpha)^{-1} h_\Lambda(y_1, \dots, y_n)$$

as desired. \square

Note that since $L \subseteq L^\circ$, Proposition 7.5 shows that $V'_{-\beta+L, \mathbb{Z}}$ is generally larger than $V_{\beta+L, \mathbb{Z}}$. We record the case $\beta = 0$ of Proposition 7.5:

Corollary 7.6. *The integral form $V'_{L, \mathbb{Z}}$ is the integral span of coefficients of products of the form*

$$E^-(-\beta_1, x_1) \cdots E^-(-\beta_k, x_k) \iota(e_\alpha)$$

where $\beta_i \in L^\circ$ and $\alpha \in L$.

Remark 7.7. Corollary 7.6 provides an alternative to the description of $V'_{L, \mathbb{Z}}$ given in Proposition 3.6 of [DG].

Remark 7.8. Note that the precise identity of $V'_{-\beta+L, \mathbb{Z}}$ depends on the choice of normalization of the invariant bilinear pairing between $V_{\beta+L}$ and $V_{-\beta+L}$. We shall take $V'_{-\beta+L, \mathbb{Z}}$ as described in Proposition 7.5 as the official $V'_{-\beta+L, \mathbb{Z}}$, since we have $V_{\beta+L, \mathbb{Z}} \subseteq V'_{-\beta+L, \mathbb{Z}}$ this way.

Finally we can classify some integral intertwining operators:

Theorem 7.9. *For any $\beta, \gamma \in L^\circ$, there is a rank one lattice of intertwining operators within $V_{\beta\gamma}^{\beta+\gamma}$ integral with respect to $V_{\beta+L, \mathbb{Z}}$, $V_{\gamma+L, \mathbb{Z}}$, and $V'_{-\beta-\gamma+L, \mathbb{Z}}$. Moreover, this lattice is spanned by*

$$\mathcal{Y}_{\beta, \gamma, \mathbb{Z}} = e^{-\pi i \langle \beta, \gamma \rangle} \varepsilon(\gamma, \beta)^{-1} \mathcal{Y}_\beta|_{V_{\beta+L} \otimes V_{\gamma+L}}.$$

Proof. From the definition (6.3) of \mathcal{Y}_β , we have

$$\begin{aligned}
\mathcal{Y}_\beta(\iota(e_\beta), x) \iota(e_\gamma) &= E^-(-\beta, x) E^+(-\beta, x) x^{\langle \beta, \gamma \rangle} e^{\pi i \langle \beta, \gamma \rangle} c(\gamma, \beta) \iota(e_\beta e_\gamma) \\
&= x^{\langle \beta, \gamma \rangle} E^-(-\beta, x) e^{\pi i \langle \beta, \gamma \rangle} \varepsilon(\gamma, \beta) \varepsilon(\beta, \gamma)^{-1} \varepsilon(\beta, \gamma) \iota(e_{\beta+\gamma}) \\
&= x^{\langle \beta, \gamma \rangle} E^-(-\beta, x) e^{\pi i \langle \beta, \gamma \rangle} \varepsilon(\gamma, \beta) \iota(e_{\beta+\gamma}).
\end{aligned}$$

Then

$$\mathcal{Y}_{\beta,\gamma,\mathbb{Z}}(\iota(e_\beta), x)\iota(e_\gamma) = x^{\langle\beta,\gamma\rangle} E^-(-\beta, x)\iota(e_{\beta+\gamma}) \in V'_{-\beta-\gamma+L,\mathbb{Z}}\{x\} \quad (7.9)$$

by Proposition 7.5.

Since $\iota(e_\beta)$ and $\iota(e_\gamma)$ generate $V_{\beta+L,\mathbb{Z}}$ and $V_{\gamma+L,\mathbb{Z}}$, respectively, as $V_{L,\mathbb{Z}}$ -modules, Theorem 3.9 implies that $\mathcal{Y}_{\beta,\gamma,\mathbb{Z}}$ is integral with respect to $V_{\beta+L,\mathbb{Z}}$, $V_{\gamma+L,\mathbb{Z}}$, and $V'_{-\beta-\gamma+L,\mathbb{Z}}$. Moreover, we see from Proposition 7.5 and (7.9) that for $c \in \mathbb{C}$,

$$c\mathcal{Y}_{\beta,\gamma,\mathbb{Z}}(\iota(e_\beta), x)\iota(e_\gamma) \in V'_{-\beta-\gamma+L,\mathbb{Z}}\{x\}$$

if and only if $c \in \mathbb{Z}$. Thus $\mathcal{Y}_{\beta,\gamma,\mathbb{Z}}$ spans the lattice of intertwining operators in $V_{\beta\gamma}^{\beta+\gamma}$ which are integral with respect to $V_{\beta+L,\mathbb{Z}}$, $V_{\gamma+L,\mathbb{Z}}$, and $V'_{-\beta-\gamma+L,\mathbb{Z}}$. \square

Remark 7.10. Note that to prove Theorem 7.9 we only need the “easy” half of Proposition 7.5 which states that for any $\beta \in L^\circ$, the coefficients of products as in (7.2) are contained in $V'_{-\beta+L,\mathbb{Z}}$. But the more difficult opposite inclusion gives the additional information that for $\beta, \gamma \in L^\circ$, the coefficients of $\mathcal{Y}_{\beta,\gamma,\mathbb{Z}}(w_\beta, x)w_\gamma$ for any $w_\beta \in V_{\beta+L,\mathbb{Z}}$, $w_\gamma \in V_{\gamma+L,\mathbb{Z}}$ are integral linear combinations of the coefficients of products as in (7.2) (with β replaced by $\beta + \gamma$).

Remark 7.11. Note that (7.9) shows that for $\beta, \gamma \in L^\circ$, $\mathcal{Y}_{\beta,\gamma,\mathbb{Z}}(\iota(e_\beta), x)\iota(e_\gamma) \notin V_{\beta+\gamma+L,\mathbb{Z}}\{x\}$ since in general $\beta \notin L$. It is not clear that any non-zero integer multiple of $\mathcal{Y}_{\beta,\gamma,\mathbb{Z}}$ is in general integral with respect to $V_{\beta+L,\mathbb{Z}}$, $V_{\gamma+L,\mathbb{Z}}$, and $V_{\beta+\gamma+L,\mathbb{Z}}$.

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